

# Mixed-asset portfolio allocation under mean-reverting asset returns

Charles-Olivier Amédée-Manesme\*    Fabrice Barthélémy †  
Philippe Bertrand ‡    Jean-Luc Prigent§

## Abstract

Standard results about portfolio optimization suggest that the allocation to real estate in a mixed-asset portfolio should be around 15 – 20%. However, the institutional investors share in real estate is significantly smaller, around 7 – 9%. Many researches have addressed this point even if as of today no consensus has emerged. In this paper, we built-up an allocation model that can explain the empirical observed weights. For this purpose, we account for the term structure of all standard financial assets and also of real estate asset class (expected returns, volatilities and correlations depending on the time to maturity). We propose a dynamic portfolio optimization model that allows analyzing portfolio weights with respect to the whole term structure modelling, due to its tractability and its good fit when being adequately calibrated. In this framework, we provide explicit and operational solutions to the dynamic mixed-asset portfolio allocation (cash, real estate, stock and bond). The results show that accounting for investment horizon and mean-reverting dynamics allows to better examine how portfolio allocations depend on both risk aversion and investment horizon.

**Keywords:** Portfolio allocation; Mixed-asset; Real estate investment; Mean reverting effects.

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\*Laval University, Department of Finance, Insurance and Real Estate, Pavillon Palasis Prince, G1V 0A6, Québec, QC, Canada, e-mail: charles-olivier.amedee-manesme@fsa.ulaval.ca

†CEMOTEV, University of Versailles, Saint-Quentin-en-Yvelines, 78047, Guyancourt, Cedex 33, France, e-mail: fabrice.barthelemy@uvsq.fr.

‡Aix-Marseille Graduate School of management - IAE, CERGAM and AMSE, Chemin de la Quille - Puyricard, CS 30063 - 13089 Aix en Provence Cedex 2 and KEDGE BS, Marseille, France, e-mail: philippe.bertrand@univ-amu.fr

§University of Cergy-Pontoise, THEMA and Labex MME-DII, 33 Bd du Port, 95011, Cergy-Pontoise, France, e-mail: jean-luc.prigent@u-cergy.fr.

# 1 Introduction

Mixed asset portfolio allocation consists usually in determining optimal weights invested on cash, bonds, stocks and finally real estate assets. For both institutional investors and academic researchers, it has been a very important topic from many years (see e.g. Webb *et al.*, 1986, 1987; Gold, 1986) and is still a rather involved problem. Indeed, the determination of the optimal allocation on real estate asset does not rise to a clear consensus, even almost all investors agree that real estate can be an effective portfolio diversifier (see e.g. Hoesli *et al.*, 2004). Hoesli and MacGregor (2000) suggest that allocation on real estate would be equal to about 15-20%. However, for instance, Fogler (1984) wonders whether we can justify a 20% weight invested on real estate. Hudson-Wilson *et al.* (2005) also deal with the rationality of real estate investment. Indeed, in many countries, looking at the portfolios of institutional investors, the weight invested on real estate is significantly smaller than 15-20%, as emphasized by Clayton (2007) and J.P. Morgan (2007) who show that the allocation on real estate is about 7.3% for the US and about 8.5% for the UK. Andonov *et al.* (2013) also find that the allocations on stocks and bonds dominate the others (about 75%) but the investment on alternative assets including real estate has increased over time.<sup>1</sup>

Summarizing results in the literature about mixed-asset investment, Seiler *et al.* (1999) show that real estate diversification ranges from 0% to 67%. Considering both public and private real estate in the mixed-asset framework (depending on the direct real estate index chosen), Feldman (2003) also finds that global real estate allocation ranges from 0% to 42%. Looking at the European financial market, Fugazza *et al.* (2007) determine the optimal weights for risk-averse investors, finding that the introduction of real estate assets in optimal portfolios yields to weights lying between 12% and 44%. They emphasize that "the welfare costs of either ignoring predictability or restricting portfolio choices to traditional financial assets only are found to be in the order of 150-300 basis points per year". According to Rehring (2012), the difference between theoretical weights and the low allocations to real estate in portfolios of institutional investors is viewed as a puzzle in real estate research (see Chun *et al.*, 2004). Lin and Liu (2008) point out also the heterogeneity of investors facing real estate returns and risk.

As regards portfolio optimization, many empirical studies have demonstrated how allocations between stocks, bonds and cash depend highly on risk aversion. In particular, bond/stock ratios differ for conservative, moderate or aggressive investors (see e.g. Brennan and Xia, 2000. de Palma and Prigent? 200). Approaches to the portfolio allocation problem are usually based on the standard

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<sup>1</sup>Andonov *et al.* (2013) examine about 880 pension funds in the United States, Canada, Europe, and Australia/New Zealand from 1990 to 2009. In 2009, the shares of these pension funds are respectively equal to 47.1% on stocks, to 36.9% on bonds, and to 2.5% on cash while the remaining amount (i.e. 13.5%) has been invested in alternative assets, especially 5.1% on Real Estate.

model of Markowitz (1952) where the usual portfolio maturities are monthly, quarterly or annual returns. Under specific assumptions (markets without friction, independent and identically distributed (i.i.d.) asset returns<sup>2</sup> and power utility to describe investor's utility), the optimal portfolio allocation is myopic. It means that, whatever the investment horizon, the weights are constant, implying that short-term and long-term asset allocations are equal. However, it is usually recognized that investors with longer investment horizon invest a higher percentage of their portfolio in stocks, which is not consistent with portfolio allocations determined from basic models of portfolio optimization (see e.g. Canner *et al.*, 1997). As emphasized by Bajoux-Besnainou *et al.* (2001), popular investment advice does not conform to the myopic property.

Additionally, the optimal portfolio allocation may depend significantly on the predictability of asset returns through their autocorrelations as emphasized for instance by Campbell and Viceira (2002). But the prediction of asset returns is difficult, especially for long time horizons (for example, are equity returns riskier than bond returns?). For the real estate market, MacKinnon and Al Zaman (2009) examine the predictability of real estate returns in the U.S. They show that direct real estate returns exhibit mean reversion process and highlight how real estate investment risk is weaker for long-term investors. Introducing investment horizons in a buy and hold strategy as in Campbell and Viceira (2005), they conclude that real estate allocation increases in the long term due to the decline of long term correlations:<sup>3</sup> “real estate is a better diversifier for long-term portfolios than for short-term”. In particular, they find (and use) a decreasing volatility over time and suggest an optimal allocation to real estate between 31% (versus 20% for short-term horizon, e.g. 1-year). Pagliari (2011) also shows how varying correlation between different asset classes modifies the optimal mixed allocation (see also Rab and Warnung, 2012). Adding the transaction costs and the marketing period risk to the return predictability,<sup>4</sup> Rehring (2012) finds the same conclusions for the U.K. commercial real estate: the allocation to real estate increases strongly with the investment horizon. The importance of return predictability for long-term horizons and of transaction costs for short and medium term is particularly highlighted. The author also finds a decreasing volatility of real estate returns over time.<sup>5</sup> The allocation to real

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<sup>2</sup>Note that most studies relying on traditional mean/variance optimization assume i.i.d. returns.

<sup>3</sup>Heaney and Srianthakumar (2012) highlight in particular how the correlation between real estate returns and capital markets is time-varying.

<sup>4</sup>Indeed, real estate markets present many specificities such as transaction costs, lack of liquidity (as illustrated - among other things - by the marketing period risk and the rental vacancy rate). It is also well documented that (the observed or reported) direct (private) real estate returns exhibit autocorrelation (see, among others, Geltner (1991) who deal with smoothing issue and Barkham and Geltner (1995) who point out that returns are not i.i.d.).

<sup>5</sup>For example, as noted by Rehring (2012): “high transaction costs imply expected real estate returns, per period, that are much higher in the long run than in the short run.” Regarding the volatility, Rehring (2012) examines the U.K. real estate market and shows that the conditional standard deviation of commercial real estate returns depends on the investment maturity as for usual stocks. In other words, Rehring (2012) internalizes the transaction costs both in the return and volatility modelling.

estate is found to increase substantially with the investment horizon (from 58% to 87% for 20 years horizon). Pagliari (2017) examines real estate’s role in institutional mixed-asset portfolios using both private and public real estate indices. He emphasizes the role of the auto-correlation of private-market asset returns. Pagliari (2017) shows that, compared to public-market asset classes that look like random walks, the annualized long-horizon volatility decreases less slowly and long-horizon correlation with most public-market assets increases. Finally, he uses a decreasing long-horizon volatility and suggests an allocation to real estate of somehow 10% to 15% for long term investors.<sup>6</sup> Therefore, as emphasized by Campbell and Viceira (2005) in the general case<sup>7</sup> and by MacKinnon and Al Zaman (2009) in the real estate context, it is necessary to estimate and to take account of the mean-reverting term-structure of expected returns and volatility risks for stocks, bonds, cash and real estate assets.

In this paper, following the aforementioned authors, we analyze the mixed-asset portfolio problem based on four basic financial assets: a money market account (the cash), a bond with constant maturity, a real estate asset and a financial stock index. We assume that the investor seeks to maximize the expected utility of her terminal wealth. According to previous comments on the markets properties and investors behaviours towards risk, we take account of various markets features as follows: First, we do no longer consider static investment strategies (i.e. "buy and hold") but instead dynamic strategies. Indeed, dynamic portfolio optimization ("continuous-time rebalancing") allows to better take account of the financial structure, in particular of the information flows; second, such modelling enables to consider different forms of term structures, for example, a mean-reverting term structure for both excess expected return and volatility; third, we consider a bond with constant maturity.<sup>8</sup>

In this framework, we contribute by providing a general solution to the portfolio allocation problem taking account on one hand of the utility of investors (and thus their risk aversion) and of their time horizons and, on the other hand, by taking account of mean-reverting properties for both expected returns and volatilities term structures. As in Wachter (2002), we assume the market completeness, but to take account simultaneously of the mean-reverting properties of expected returns, volatilities and correlations, we introduce deterministic time-varying functions to model both excess returns and volatilities. As shown in the empirical and numerical sections, such functions correspond usually to negative exponentials with respect to time.<sup>9</sup>

<sup>6</sup>All the previous authors find decreasing volatilities function for real estate and thus real estate is less risky for longer-horizon investors. Similar results are provided by Baroni *et al.* (2008) for the French residential market in Paris.

<sup>7</sup>Campbell and Viceira (2005) examine the U.S. market and show that, contrary to cash, stock returns are mean reverting. This means that the long-term volatility of stock returns (per period) is lower than the short-term return volatility. They find also that bond returns are slightly mean reverting.

<sup>8</sup>As discussed by Bajoux-Besnainou *et al.* (2001), the introduction of constant duration bonds allows to get a bond/stock ratio which increases with time when assuming that returns follow geometric Brownian motions.

<sup>9</sup>Wachter (2002) considers only mean-reverting property of the drift of one single as-

The intertemporal optimization problem is solved by using the martingale approach. To model the multifactor term structure avoiding any arbitrage opportunity, we extend the model introduced by Chiarella *et al.* (2016). Using the martingale approach developed by Cox and Huang (1989), we provide explicit solutions for the optimal portfolio values and the associated portfolio weights. We detail in particular the logarithm, CARA and CRRA cases. Such approach leads to more realistic results. To illustrate these features, we conduct a numerical analysis from Rehring (2012) estimates. The paper is organized as follows. First, we introduce the financial market modelling. In particular, we detail the multi factor model that describes the bond and the real estate asset dynamics precluding any arbitrage opportunity. We then explicitly compute the risk-neutral density and determine all the asset risk premia. Then, we provide the solution of the optimization problem, for quite general utility functions and especially for the logarithm, CARA and CRRA cases. Finally, we illustrate the mean-reversion properties of assets and compute numerically the solution for the CRRA case. As a by-product, we show how to calibrate our continuous-time model to cumulative data over time as those of Rehring (2012). Some of the technical proofs are relegated to appendices.

## 2 The Financial Market

To take account of various markets features as mentioned previously, we first generalize the framework considered in Chiarella *et al.* (2016) by introducing a multifactor term structure model with time-varying drifts and volatilities precluding any arbitrage opportunity. It is a generalization of Black and Scholes model and a variant of the Merton (1971) one state variable model. The model we consider assumes normality of log returns. This assumption is not too restrictive, when dealing with long term investment. This allows in particular getting explicit formulas. Since standard bonds with long maturities are not generally available on the financial markets and to be in accordance with popular advice, we assume that the bond has a constant duration, as in Bajoux-Besnainou *et al.* (2001) and Rehring (2012). Note that this general modelling with time-varying drifts and volatilities allows to calibrate the model in order to take account of the mean-reverting properties as documented by Rehring (2012) and Pagliari (2017) (see Appendix C).

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set. This paper introduces a specific Ornstein-Uhlenbeck process to model the instantaneous Sharpe ratio (but with a constant volatility) which is driven by the same Brownian motion as for the dynamics of the risky asset. This allows the financial market completeness. In a discrete-time setting, Campbell and Viceira (2005) introduce a vector autoregressive (VAR) model, which justifies the mean-reverting property of the drift. However, such approach does not lead to exact explicit solutions and moreover volatility is also assumed to be constant. They deal also with only one single asset. When dealing with multi asset allocation, time-varying correlation must be also taken into account. Additionally, looking at financial data, return volatilities are also mean reverting. Finally, in the numerical section, note that all excess expected returns per year and standard deviations per year look like negative exponential functions of time to maturity. Thus, they correspond to the expectations of stochastic mean-reverting processes such as the Ornstein-Uhlenbeck process.

## 2.1 The set of basic assets

The market is assumed to be arbitrage-free and without friction. Financial transactions occur in continuous-time, along a time period  $[0, T]$ . Four basic assets are available at any time on the market:

- An instantaneously riskless money market fund, the cash, with a price denoted by  $C$ ;
- A real estate asset  $P$ ;
- A stock index fund with a price  $S$ ;
- A bond fund  $B$  with constant duration  $D_B$  obtained by rolling continuously bonds throughout the investment period  $[0, T]$ . It is denoted by  $B_D$  which is a zero-coupon bond with maturity  $(t + D)$  at time  $t$ .

Since continuous-time rebalancing is allowed, financial markets can be assumed to be complete by introducing three sources of risk. For this purpose, we introduce a multidimensional Brownian motion  $W = (W^r, W^P, W^S)$  to describe the uncertainty of the asset returns. The correlation matrix is given by:

$$\Sigma_c = \begin{bmatrix} 1 & \rho^{r,P} & \rho^{r,S} \\ \rho^{r,P} & 1 & \rho^{P,S} \\ \rho^{r,S} & \rho^{P,S} & 1 \end{bmatrix}. \quad (1)$$

Recall that the predictable compensator  $\langle W^a, W^b \rangle$  of the product  $W^a W^b$  satisfies (see e.g. Jacod and Shiryaev, 2002):

$$d\langle W^a, W^b \rangle_t = \rho^{a,b} dt.$$

## 2.2 Basic asset dynamics

- The money market  $C$  satisfies:

$$\frac{dC_t}{C_t} = r_t dt. \quad (2)$$

- The bond fund  $B_D$  with constant duration  $D_B$  is solution of:

$$\frac{dB_{D_B,t}}{B_{D_B,t}} = (r_t + \theta_t^B(D_B))dt - \beta_r(D_B) a_r dW_t^r, \quad (3)$$

where  $\theta^B(D_B)$  is the risk premium of bond  $B$ .

- The real estate asset  $P$  satisfies:

$$\frac{dP_{D_P,t}}{P_{D_P,t}} = (r_t + \theta_t^P)dt + \sigma_t^P dW_t^P, \quad (4)$$

- The stock index fund with price  $S$  is defined by:

$$\frac{dS_t}{S_t} = (r_t + \theta_t^S)dt + \sigma_t^S dW_t^S. \quad (5)$$

Denote by  $\tau$  the remaining time to maturity  $T$  (i.e.  $\tau = T - t$ ). We assume that the real interest rate  $r_t$  follows an Ornstein-Uhlenbeck process given by:

$$dr_t = k_r(\bar{r} - r_t)dt + a_r dW_t^r, \quad (6)$$

where the convergence speed  $k_r$  and the long term value  $\bar{r}$  are positive constants.<sup>10</sup>

The bond pricing formula is based on an exponential affine model, as introduced by Duffie and Kan (1996):

$$B(r_t, t, T) = \exp[-\alpha(\tau) - \beta_r(\tau)r_t], \quad (7)$$

where  $\alpha(\tau)$  and  $\beta_r(\tau)$  are determined by using the no-arbitrage condition. Due to the normalization, the coefficients satisfy the following terminal conditions at maturity:

$$\alpha(0) = 0 \text{ and } \beta_r(0) = 0.$$

We have:

$$\frac{dB(r_t, t, T)}{B(r_t, t, T)} = \mu^B(t, \tau)dt - \beta_r(\tau)a_r dW_t^r,$$

$$\mu^B(t, \tau) = \alpha'(\tau) + \beta_r'(\tau)r_t - \beta_r(\tau)k_r(\bar{r} - r_t) + \frac{1}{2} [\beta_r^2(\tau)a_r^2].$$

$$\theta_t^B = \alpha'(\tau) + \beta_r'(\tau)r_t - \beta_r(\tau)k_r(\bar{r} - r_t) + \frac{1}{2} [\beta_r^2(\tau)a_r^2] - r_t$$

$$\sigma_t^B = \beta_r(T - t)a_r$$

In order to obtain the bond price, we use the standard no-arbitrage argument as in Chiarella *et al.* (2016). The financial market is complete. Therefore, there exists a unique risk-neutral probability  $\mathbb{Q}$  associated to three market premia,  $\lambda_r$ ,  $\lambda_P$  and  $\lambda_S$  which density  $\eta$  with respect to the initial probability  $\mathbb{P}$  is given by:

$$\eta_t = \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathbf{F}_t \right] = \exp \left[ -\mathbf{M}_t - \int_0^t A_s ds \right], \quad (8)$$

where

$$\mathbf{M}_t = \int_0^t \lambda_{r,s} dW_s^r + \int_0^t \lambda_{P,s} dW_s^P + \int_0^t \lambda_{S,s} dW_s^S, \quad (9)$$

and

$$A_t = \frac{1}{2} (\lambda_{r,t}^2 + \lambda_{P,t}^2 + \lambda_{S,t}^2) + \lambda_{r,t} \lambda_{P,t} \rho^{r,P} + \lambda_{r,t} \lambda_{S,t} \rho^{r,S} + \lambda_{P,t} \lambda_{S,t} \rho^{P,S}. \quad (10)$$

<sup>10</sup>Recall that the Ornstein-Uhlenbeck process is given by:

$$r_t = r_0 e^{-k_r t} + \bar{r}(1 - e^{-k_r t}) + a_r e^{-k_r t} \int_0^t e^{k_r s} dW_s^r.$$

It has been introduced by Vasicek (1977) to model stochastic interest rates (see e.g. Brigo and Mercurio, 2006).

Each of the three basic assets  $B$ ,  $P$  and  $S$  must satisfy the following condition: when they are discounted by the nominal money market account  $C$ , they must be martingales with respect to the risk-neutral probability  $\mathbb{Q}$ . This is equivalent to the fact that, when they are multiplied by the Radon-Nikodym density and divided by  $C$ , they must be martingales with respect to the historical probability  $\mathbb{P}$ . It is equivalent to the fact that their bounded variation components are equal to 0. This later condition implies the three following equalities:

Condition 1:  $((B(r_t, t, T)/\exp[\int_0^t r_s ds]) \times \eta_t)_t$  is a  $\mathbb{P}$ -martingale:

$$\mu^B(t, \tau) - r_t + a_r \beta_r(\tau) [\lambda_{r,t} + \lambda_{P,t} \rho^{r,P} + \lambda_{S,t} \rho^{r,S}] = 0. \quad (11)$$

Condition 2:  $((P_t/\exp[\int_0^t r_s ds]) \times \eta_t)_t$  is a  $\mathbb{P}$ -martingale:

$$\mu_t^P - r_t - \sigma_t^P (\lambda_{r,t} \rho^{r,P} + \lambda_{P,t} + \lambda_{S,t} \rho^{P,S}) = 0. \quad (12)$$

Condition 3:  $((S_t/\exp[\int_0^t r_s ds]) \times \eta_t)_t$  is a  $\mathbb{P}$ -martingale:

$$\mu_t^S - r_t - \sigma_t^S [\lambda_{r,t} \rho^{r,S} + \lambda_{P,t} \rho^{P,S} + \lambda_{S,t}] = 0. \quad (13)$$

From the previous system of equations, we deduce the risk premia of the three basic assets  $P$ ,  $B$ , and  $S$ . They are defined as the excess instantaneous expected returns with respect to the interest rate  $r$ .

**Proposition 1** *The three risk premia  $\theta^P$ ,  $\theta^B$  and  $\theta^S$  are given by:*

*i) Nominal Bond B: For fixed duration  $D_B$ ,*

$$\theta_t^B = \mu^B(t, D_B) - r_t = -a_r \beta_r(D_B) [\lambda_{r,t} + \lambda_{P,t} \rho^{r,P} + \lambda_{S,t} \rho^{r,S}]. \quad (14)$$

*ii) Real estate asset P:*

$$\theta_t^P = \mu_t^P - r_t = \sigma_t^P (\lambda_{r,t} \rho^{r,P} + \lambda_{P,t} + \lambda_{S,t} \rho^{P,S}). \quad (15)$$

*iii) Stock price S:*

$$\theta_t^S = \mu_t^S - r_t = \sigma_t^S [\lambda_{r,t} \rho^{r,S} + \lambda_{P,t} \rho^{P,S} + \lambda_{S,t}], \quad (16)$$

where  $\lambda^r$ ,  $\lambda^P$  and  $\lambda^S$  are usually interpreted as the market prices of risk (or factor loadings) associated respectively to the sources of risk  $W^r$ ,  $W^P$  and  $W^S$ .

Functions  $\alpha(\cdot)$  and  $\beta_r(\cdot)$ , which are involved in the exponential affine model describe the value of the bond. They are given by:

$$\beta_r(\tau) = \left( \frac{1 - \exp(-\tau k_r)}{k_r} \right).$$

and

$$\frac{\alpha(\tau)}{\tau} = a_r \left[ \frac{\bar{r}}{a_r} - \frac{\lambda_r + \lambda_P \rho^{r,P} + \rho^{r,S} \lambda_S}{k_r} \right] \left[ 1 - \frac{1}{\tau k_r} + \frac{\exp(-\tau k_r)}{\tau k_r} \right]$$

$$-\frac{a_r^2}{2k_r^2} \left[ 1 - \frac{2[1 - \exp(-\tau k_r)]}{\tau k_r} + \frac{1 - \exp(-2\tau k_r)}{2\tau k_r} \right].$$

In this setting,  $B(r_t, t, T)$  can be replicated using the assets  $C$  and  $B_D$  which span the bond market, as in Bajeux *et al.* (2001) for a simpler case. This dynamic combination of fixed-income securities of different durations is referred to as the *passive immunization* (see Fong, 1990; Fabozzi, 2000). From previous Proposition 1, we can determine the market prices of risk  $\lambda_r$ ,  $\lambda_P$  and  $\lambda_S$ . Indeed, introduce the matrix  $\Gamma$  equal to:

$$\begin{bmatrix} -a_r \beta_r(D_B) & -a_r \beta_r(D_B) \rho^{r,P} & -a_r \beta_r(D_B) \rho^{r,S} \\ \sigma_t^P \rho^{r,P} & \sigma_t^P & \sigma_t^P \rho^{P,S} \\ \sigma_t^S \rho^{r,S} & \sigma_t^S \rho^{P,S} & \sigma_t^S \end{bmatrix}. \quad (17)$$

We have:

$$\begin{bmatrix} \theta^B \\ \theta^P \\ \theta^S \end{bmatrix} = \Gamma \cdot \begin{bmatrix} \lambda_r \\ \lambda_P \\ \lambda_S \end{bmatrix}, \text{ which implies } \begin{bmatrix} \lambda_r \\ \lambda_P \\ \lambda_S \end{bmatrix} = \Gamma^{-1} \cdot \begin{bmatrix} \theta^B \\ \theta^P \\ \theta^S \end{bmatrix}.$$

### 3 Optimal portfolios

Let us introduce the matrix  $\Sigma$  of the rates of return of  $B$ ,  $P$ ,  $S$  given by:

$$\Sigma_t = \begin{bmatrix} -\beta_r(D_B) a_r & 0 & 0 \\ 0 & \sigma_t^P & 0 \\ 0 & 0 & \sigma_t^S \end{bmatrix}. \quad (18)$$

We recall the standard results about optimal portfolio computation when the financial is complete (see e.g. Prigent, 2007). Portfolio weights are respectively denoted by  $x_C$ ,  $x_B$ ,  $x_P$  and  $x_S$  (with  $x_C + x_B + x_P + x_S = 1$ ). The portfolio value at time  $t$  is denoted by  $V_t$ . Therefore, the portfolio value  $V$  follows the dynamics:

$$\begin{aligned} \frac{dV_t}{V_t} &= [r_t + x_B(t)\theta_t^B + x_P(t)\theta_t^P + x_S(t)\theta_t^S] dt \\ &+ x_B(t) [-\beta_r(D_B) a_r dW_t^r] + x_P(t) [\sigma_t^P dW_t^P] + x_S(t) [\sigma_t^S dW_t^S]. \end{aligned} \quad (19)$$

The investor's preferences is described by her utility function  $U$ , which embeds her risk aversion. Function  $U$  is assumed to satisfy usual properties, namely it is strictly increasing, strictly concave and twice continuously differentiable (see e.g. Gollier (2001) for definitions and main properties of utility functions). We consider an investor with an initial capital denoted by  $V_0$ . She is assumed to maximize the expected utility over the time horizon  $T$ , defined on the portfolio value  $V_t$ . Thus, her optimal portfolio weights are the solutions of the following problem:

$$\underset{x_C, x_B, x_P, x_S}{Max} \mathbb{E}[U(V_T)].$$

For different utility functions, we determine the optimal portfolio. We show in particular how it depends on the investor's risk aversion. We detail the solutions for the logarithm, HARA and CARA utility functions.

### 3.1 General result

To solve the optimization problem given  $V_0$ , where the set of variables is the collection of all self-financing strategies, Cox and Huang (1989) use the market completeness to propose a change of variables. The new variables are all possible portfolio values at maturity with only one constraint:

$$V_0 = \mathbb{E}_{\mathbb{P}} \left[ V_T \frac{\eta_T}{\exp \left( \int_0^T r_s ds \right)} \right]. \quad (20)$$

Differentiating with respect to the variables  $V_T(\omega)$  for each random event  $\omega$ , leads to the equation:

$$U'(V_T) = \nu \frac{\eta_T}{\exp \left( \int_0^T r_s ds \right)},$$

where  $\nu$  is a Lagrange multiplier determined by the initial investment constraint. Denoting  $J(y) = (U')^{-1}(y)$ , the optimal solution is given by:

$$V_T^* = J \left( \nu \frac{1}{H_T} \right). \quad (21)$$

where  $H_T$  is the numeraire portfolio, equal to:

$$H_T = \left( \frac{\eta_T}{\exp \left( \int_0^T r_s ds \right)} \right)^{-1}. \quad (22)$$

To compute the replicating strategies for a given optimal portfolio, first note that  $V_t^*/H_t$  is a martingale. Thus, we obtain the martingality relation:

$$\frac{V_t^*}{H_t} = \mathbb{E}_{\mathbb{P},t} [V_T^*/H_T]. \quad (23)$$

Therefore, it is necessary to compute conditional expectations of quantities which are functions of  $V_t^*/H_t$ . For this purpose, the conditional expectations of the numeraire portfolio have to be used. We first consider the logarithm utility case. For the other cases, note that the optimal portfolios are functions of the optimal logarithm portfolio.

#### 3.1.1 The logarithm case

The logarithm utility function is defined by  $U(x) = \ln(x)$  for  $x > 0$ , so that the absolute risk aversion is  $-U''(x)/U'(x) = 1/x$  and the relative risk aversion is

constant and equal to  $-xU''(x)/U'(x) = 1$ . In that case, the optimal portfolio is called the numeraire portfolio (see Long (1990) or the growth-optimal portfolio (see Merton, 1992). Its value at maturity  $T$ , denoted by  $V_T^{\text{ln}}$ , is given by:

$$V_T^{\text{ln}} = V_0 H_T. \quad (24)$$

Note that  $\left(\int_t^T r_s ds\right)$  has a Gaussian distribution. Thus, the ratio

$$[(H_T/H_t)]^z$$

has a Lognormal distribution for any power  $z \neq 0$ . This ratio is equal to  $\exp[N_z(t, T)]$  where  $N_z(t, T)$  has a Gaussian distribution. Therefore, the conditional expectation  $E_t[(H_T/H_t)]^z$  is defined by:

$$\mathbb{E}_t[(H_T/H_t)]^z = \exp[E(N_z)(t, T) + \frac{1}{2}\text{Var}(N_z)(t, T)], \quad (25)$$

where: (see Appendix A)

$$\mathbb{E}(N_z)(t, T) = z\Phi_{(t, T)} \text{ and } \text{Var}(N_z)(t, T) = z^2\Psi_{(t, T)},$$

$$\Phi_{(t, T)} = (T - t) \times \left( \bar{r} + (r_t - \bar{r}) \frac{\beta_r(T - t)}{T - t} + \frac{1}{T - t} \int_t^T A(s) ds \right),$$

and

$$\begin{aligned} \Psi_{(t, T)} = & \frac{a_r^2}{k_r^2} \left[ (T - t) + \frac{1}{2k_r} [1 - \exp(-2(T - t)k_r)] - 2\beta_r(T - t) \right] \\ & + 2 \int_t^T A(s) ds + 2a_r \int_t^T (\lambda_{r,s} + \lambda_{P,s}\rho^{r,P} + \lambda_{S,s}\rho^{r,S}) \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds, \end{aligned}$$

with

$$A(t) = \frac{1}{2} (\lambda_{r,t}^2 + \lambda_{P,t}^2 + \lambda_{S,t}^2) + \lambda_{r,t}\lambda_{P,t}\rho^{r,P} + \lambda_{r,t}\lambda_{S,t}\rho^{r,S} + \lambda_{P,t}\lambda_{S,t}\rho^{P,S}.$$

Denote respectively by  $\hat{\alpha}_{N,(t, T)}$  and  $\hat{\beta}_{N,(t, T)}$  the conditional expectations:

$$\mathbb{E}_t[(H_t/H_T)] \text{ and } \mathbb{E}_t[H_t \ln((H_t/H_T))/H_T].$$

Then we deduce:

$$\hat{\alpha}_{N(t, T)} = \exp[-\Phi_{(t, T)} + \frac{1}{2}\Psi_{(t, T)}] \text{ and } \hat{\beta}_{N(t, T)} = \hat{\alpha}_{N(t, T)} \times (\Phi_{(t, T)} - \Psi_{(t, T)}).$$

In order to simplify notations, we respectively denote by  $E(N_z)(T)$ ,  $\text{Var}(N_z)(T)$ ,  $\Phi_T$ ,  $\Psi_T$ ,  $\hat{\alpha}_{N, T}$  and  $\hat{\beta}_{N, T}$  the values of  $E_t(N_z)(0, T)$ ,  $\text{Var}(N_z)(0, T)$ ,  $\Phi_{(0, T)}$ ,  $\Psi_{(0, T)}$ ,  $\hat{\alpha}_{(0, T)}$  and  $\hat{\beta}_{(0, T)}$ .

For the logarithmic case, the optimal weights are determined as follows. The process  $\left(\frac{V^{\text{ln}}}{H}\right)_t$  is a  $\mathbb{P}$ -martingale. Therefore, we have:

$$\frac{V_t^{\text{ln}}}{H_t} = \mathbb{E}_{\mathbb{P},t} \left[ \frac{V_T}{H_T} \right],$$

which implies:

$$V_t^{\text{ln}} = V_0 H_t.$$

But, the portfolio value  $V$  is solution of the following (SDE):

$$\begin{aligned} \frac{dV_t^{\text{ln}}}{V_t} &= [r_t + x_B(t)\theta^B + x_P(t)\theta^P + x_S(t)\theta^S]dt \\ &+ x_B(t) [-\beta_r(D_B)a_r dW_t^r] + x_P(t) [\sigma_t^P dW_t^P] + x_S(t) [\sigma_t^S dW_t^S]. \end{aligned}$$

Additionally, we have:

$$dV_t^{\text{ln}} = V_0 dH_t \text{ and } dH_t = H_t [r_t dt + \mathbf{M}_t + 2A(t)dt].$$

To determine the weights, we must identify the martingale parts (the three stochastic Brownian integrals). We obtain the following system:

$$\begin{cases} \lambda_{r,t} &= -x_B(t)\beta_r(D_B)a_r \\ \lambda_{P,t} &= x_P(t)\sigma_t^P \\ \lambda_{S,t} &= x_S(t)\sigma_t^S \end{cases}$$

This relation is equivalent to:<sup>11</sup> (denote by  $\Lambda$  the factors vector and  $X$  the weighting vectors on  $B$ ,  $P$  and  $S$ )

$$\Lambda = {}^t\Sigma X, \text{ which is also equivalent to } X = ({}^t\Sigma)^{-1} \Lambda.$$

Therefore, we deduce the optimal weights at any time  $t$ .

**Proposition 2** *For the logarithmic case, the optimal weights are given by:*

$$\begin{cases} x_B(t) = h_{B,t} &= -\frac{\lambda_{r,t}}{\beta_r(D_B)a_r} \\ x_P(t) = h_{P,t} &= \frac{\lambda_{P,t}}{\sigma_t^P} \\ x_S(t) = h_{S,t} &= \frac{\lambda_{S,t}}{\sigma_t^S} \\ x_C(t) = h_{C,t} &= 1 - x_B(t) - x_P(t) - x_S(t) \end{cases} \quad (26)$$

*In the growth-optimal portfolio,  $h_C$ ,  $h_B$ ,  $h_P$  and  $h_S$  represent respectively the weights invested on the nominal money account, on the value of the bond with constant maturity, on the real asset with constant duration and on the stock.*

<sup>11</sup>Note also that we can determine the optimal weighting vector  $X$  from the vector of risk premia  $\theta$  itself since we have:

$$X = ({}^t\Sigma)^{-1} \Lambda = ({}^t\Sigma)^{-1} \Gamma^{-1} \theta.$$

### 3.1.2 CARA utility function

Assume now that the investor has a constant absolute risk aversion case which corresponds to the exponential utility function  $\widehat{U}_a(x) = -\exp(-ax) / a$  where  $a$  is the absolute risk aversion ( $a > 0$ ). The optimal portfolio value for the exponential utility function,  $V_T^{\text{exp}}$ , is a function of the numeraire portfolio given by:

$$V_T^{\text{exp}} = J\left(v \frac{1}{H_T}\right),$$

which is equivalent to:

$$V_T^{\text{exp}} = -\frac{1}{a} [\ln(\lambda) + \ln(1/H_T)] = -\frac{1}{a} \left[ \log(v) + \ln(\eta_T \exp(-\int_0^T R_s ds)) \right],$$

where  $v$  is solution of:

$$\ln(v) = \frac{-aV_0 - \mathbb{E}[(1/H_T) \ln(1/H_T)]}{\mathbb{E}[(1/H_T)]}.$$

The previous condition implies that the optimal value  $V_T^{\text{exp}}$  is equal to:

$$V_t^{\text{exp}} = H_t \mathbb{E}_t \left[ (1/H_T) \left( \frac{1}{a} A(V_0) + \ln(H_T) \right) \right],$$

where:

$$A(V_0) = \frac{aV_0 + \mathbb{E}[(1/H_T) \ln(1/H_T)]}{\mathbb{E}[(1/H_T)]} = \frac{aV_0 + \widehat{\beta}_{N,T}}{\widehat{\alpha}_{N,T}}.$$

Using the martingality relation, the value  $V_T^{\text{exp}}$  is given by:

$$V_t^{\text{exp}} = H_t \mathbb{E}_t \left[ (1/H_T) \left( \frac{1}{a} A(V_0) + \ln(H_T) \right) \right].$$

Finally, we obtain:

**Proposition 3** *For the CARA case, the optimal portfolio value at any time  $t$  of the management period is given by:*

$$V_t^{\text{exp}} = \left[ A(V_0) \widehat{\alpha}_{N,(t,T)} + \frac{1}{a} \widehat{\beta}_{N,(t,T)} \right] + \frac{1}{a} \ln(H_t) \widehat{\alpha}_{N,(t,T)}. \quad (27)$$

Using the same approach as for the numeraire portfolio, we can determine the optimal weights for the CARA case, as shown in Appendix B.

### 3.1.3 HARA utility function

Consider utility function which has a hyperbolic absolute risk aversion (HARA). This family of utility functions can be written as follows:

$$U_\gamma(x) = \left( \frac{\gamma}{1-\gamma} \right) \left( \frac{x-x^*}{\gamma} \right)^{1-\gamma}. \quad (28)$$

where  $\gamma$  and  $x^*$  are two parameters that cannot be both negative. The risk aversion is an increasing function of the absolute value of  $\gamma$ . It includes the CRRA utility function ( $x^* = 0$ ), in particular the logarithmic case for  $\gamma = 1$ . The optimal portfolio at maturity  $T, V_{(\gamma, T)}^{HARA}$ , is given by:

$$V_{(\gamma, T)}^{HARA} = x^* + \left( \frac{V_0 - x^* \hat{\alpha}_{N, T}}{E[(H_T)^{\frac{1-\gamma}{\gamma}}]} \right) (H_T)^{\frac{1}{\gamma}}. \quad (29)$$

This expression can be interpreted as follows: the optimal portfolio is a combination of a CRRA fund with  $\gamma$  parameter and a zero coupon bond with nominal  $x^*$  at time  $T$ .

We consider the special case of CRRA utility function. This means that the investor does not impose the guarantee constraint  $V_T \geq x^*$  at maturity. Thus, this case corresponds to  $x^* = 0$ .

The optimal portfolio weights for the CRRA utility function are determined from the following relations (see Appendix B for details): (denote  $z = \frac{1-\gamma}{\gamma}$ )

$$\begin{cases} \frac{\lambda_{r,t}}{\gamma} + z a_r \beta_r (T-t) & = \delta_B(t) = -x_B(t) \beta_r (D_B) a_r \\ \frac{\lambda_{P,t}}{\gamma} & = \delta_P(t) = x_P(t) \sigma_t^P \\ \frac{\lambda_{S,t}}{\gamma} & = \delta_S(t) = x_S(t) \sigma_t^S \end{cases}$$

Therefore, we deduce the optimal weights at any time  $t$ .

**Proposition 4** *The optimal weights for the CRRA case are given by:*

$$\begin{cases} x_B^{CRRA}(t) & = -\frac{\delta_B(t)}{\beta_r (D_B) a_r} \\ x_P^{CRRA}(t) & = \frac{\delta_P(t)}{\sigma_t^P} \\ x_S^{CRRA}(t) & = \frac{\delta_S(t)}{\sigma_t^S} \\ x_C^{CRRA}(t) & = 1 - x_B^{CRRA} - x_P^{CRRA} - x_S^{CRRA} \end{cases} \quad (30)$$

**Remark 5** *(Two-funds separation) The optimal CRRA weights  $X^{CRRA}$  can be decomposed as follows:*

$$X^{CRRA} = \frac{1}{\gamma} X^{Log} + \left(1 - \frac{1}{\gamma}\right) X^{Conservative}, \quad (31)$$

where  $X^{Log}$  is the optimal portfolio for the logarithm case, and  $X^{Conservative}$  corresponds to the optimal portfolio for an infinite relative risk aversion  $\gamma$ . Here, this latter portfolio is equal to:

$$X^{Conservative} = \begin{pmatrix} \beta_r (T-t) / \beta_r (D_B) \\ 0 \\ 0 \end{pmatrix}. \quad (32)$$

**Remark 6** *The ratios of asset weights are immediately deduced from Proposition 4. Indeed, for example both bond/stock and real estate/stock ratios are respectively given by:*

$$\begin{aligned} x_B(t)/x_S(t) &= h_{B,t}/h_{P,t} + (\gamma - 1)\beta_r(T - t)/(\beta_r(D_B)/h_{P,t}), \\ x_P(t)/x_S(t) &= h_{P,t}/h_{S,t}. \end{aligned} \quad (33)$$

*In Bajeux-Besnainou et al. (2001), when there is no real estate asset and when risky assets follow geometric Brownian motions, the first above equation shows that the bond/stock ratio is increasing with respect to time.*

To study the corresponding portfolio return distribution, we briefly recall how to compute its expectation and its cumulative distribution function. The portfolio return is given by:

$$V_T^{CRRRA} = \frac{V_0}{\mathbb{E} \left[ \frac{1-\gamma}{H_T^\gamma} \right]} (H_T)^\gamma.$$

The mean of the return per year is given by:

$$\mathbb{E} \left[ \left( \frac{V_T^{CRRRA}}{V_0} \right)^{\frac{1}{T}} \right] = \frac{\mathbb{E} \left[ \left[ (H_T)^\gamma \right]^{\frac{1}{T}} \right]}{\left( \mathbb{E} \left[ (H_T)^\gamma \right] \right)^{\frac{1}{T}}}.$$

But we have:

$$\mathbb{E} \left[ H_T^{\frac{1-\gamma}{\gamma}} \right] = \exp \left[ z\Phi_{(0,T)} + \frac{1}{2}z^2\Psi_{(0,T)} \right], \text{ with } z = \frac{1-\gamma}{\gamma}.$$

Consider now the real portfolio return per year  $\left( \frac{V_T^{(r)CRRRA}}{V_0} \right)^{\frac{1}{T}}$ . We get:

$$\begin{aligned} \mathbb{P} \left[ \left( \frac{V_T^{CRRRA}}{V_0} \right)^{\frac{1}{T}} \leq x \right] &= \\ \mathbb{P} \left[ \exp \left[ \frac{1}{\gamma} N_1(0,T) \right] \leq x^T \exp \left[ z\Phi_{(0,T)} + \frac{1}{2}z^2\Psi_{(0,T)} \right] \right] &= \\ \mathbb{P} \left[ N_1(0,T) \leq \gamma \left( \log(x^T) + z\Phi_{(0,T)} + \frac{1}{2}z^2\Psi_{(0,T)} \right) \right] &= \\ N \left[ \frac{\gamma T \log(x) - \gamma\Phi_{(0,T)} + \frac{1}{2} \frac{(1-\gamma)^2}{\gamma} \Psi_{(0,T)}}{\sqrt{\Psi_{(0,T)}}} \right], & \text{ with } z = \frac{1-\gamma}{\gamma}. \end{aligned}$$

The expected real return per year  $T$  is given by:

$$\mathbb{E}_0 \left[ \left( \frac{V_T^{CRRRA}}{V_0} \right)^{\frac{1}{T}} \right] = \mathbb{E}_0 \left[ \frac{(H_T)^{\frac{1}{\gamma T}}}{\mathbb{E} \left[ (H_T)^{\frac{1-\gamma}{\gamma T}} \right]} \right] = \frac{\exp \left[ \frac{1}{\gamma T} \Phi_{(0, T)} + \frac{1}{2} \left( \frac{1}{\gamma T} \right)^2 \Psi_{(0, T)} \right]}{\exp \left[ \frac{1-\gamma}{\gamma T} \Phi_{(0, T)} + \frac{1}{2} \left( \frac{1-\gamma}{\gamma T} \right)^2 \Psi_{(0, T)} \right]},$$

from which we deduce:

$$\mathbb{E}_0 \left[ \left( \frac{V_T^{CRRRA}}{V_0} \right)^{\frac{1}{T}} \right] = \exp \left[ \frac{1}{T} \Phi_{(0, T)} - \frac{1}{T^2} \left( \frac{1}{2} - \frac{1}{\gamma} \right) \Psi_{(0, T)} \right].$$

## 4 Numerical illustration

We illustrate numerically the theoretical solution from Rehring (2012), who consider annual dataset from 1965 to 2008 for the U.K. market. The cash return corresponds to U.K. three-month treasury bill, the bond yield to yield of Barclays gilt index, the stock to Barclays equity index and the real estate is constructed as described in Rehring (2012) from unsmoothed log real capital returns.

### 4.1 Excess return per year and instantaneous excess return (term structure)

Recall that the short-term interest rate is assumed to follow an Ornstein-Uhlenbeck process as introduced by Vasicek (1977) (see Relation 6). Using Rehring (2012) data on UK three-month treasury bill, the estimates of the short-term parameters are given by:

$$a_r = 0.03, k_r = 0.2; r_0 = 1.8\%; \bar{r} = 1.8\%.$$

In a first step we calibrate the excess cumulated expected returns per year.

We find that approximations by negative exponentials of type  $\alpha_\mu - \beta_\mu \exp(-\lambda_\mu t)$  provide a good fit to observed data (see Figures 1 and 3).

For the real estate  $P$  and the stock  $S$ , we get respectively:

$$\begin{aligned} \alpha_\mu^P &= 0.04; \beta_\mu^P = 0.08; \lambda_\mu^P = 0.4, \\ \alpha_\mu^S &= 0.061; \beta_\mu^S = 0.007; \lambda_\mu^S = 0.061. \end{aligned}$$

Then, to fit instantaneous excess return term structure, we begin by subtracting the expectation of the short-term interest rate, namely:

$$\mathbb{E}[r_t] = \bar{r} - (\bar{r} - r_0) \exp(-k_r t).$$

Finally, we apply results of Appendix C to identify the instantaneous excess return of both the real estate asset and the stock, as shown in Figures 1, 2, 3 and 4.

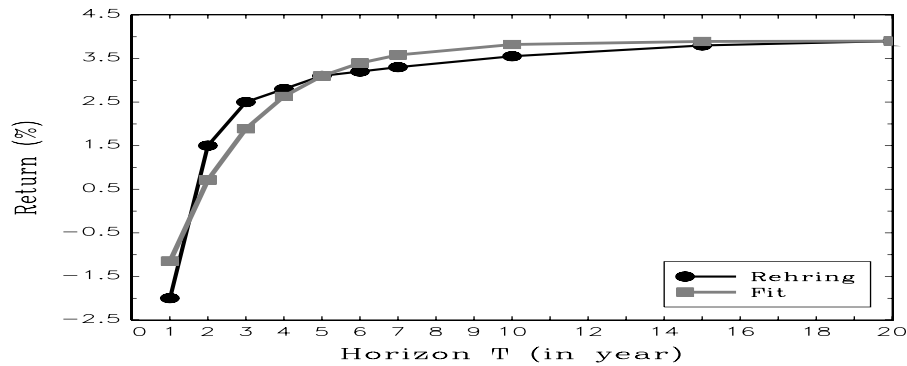


Figure 1: Calibration of the expected return per year of the real estate asset

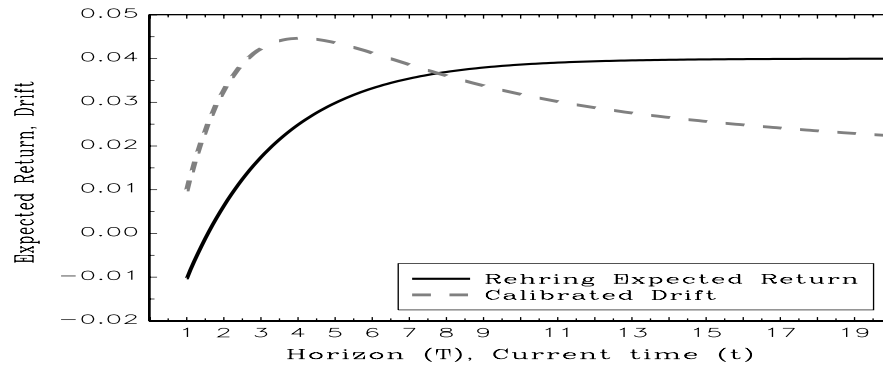


Figure 2: Calibration of the drift term structure of the real estate asset

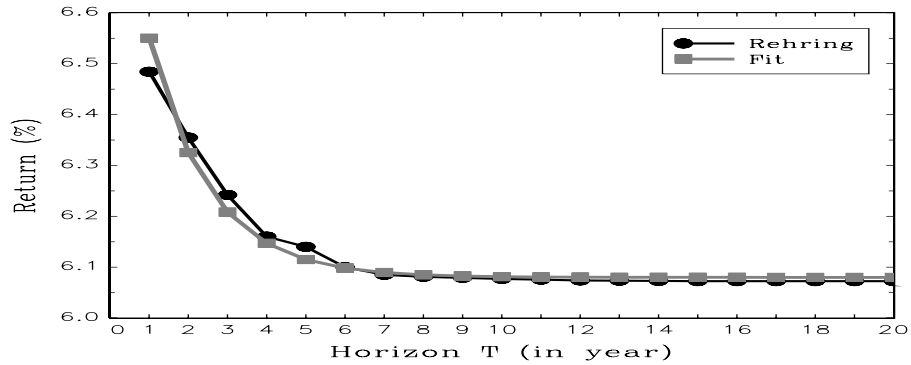


Figure 3: Calibration of the expected return per year of the stock

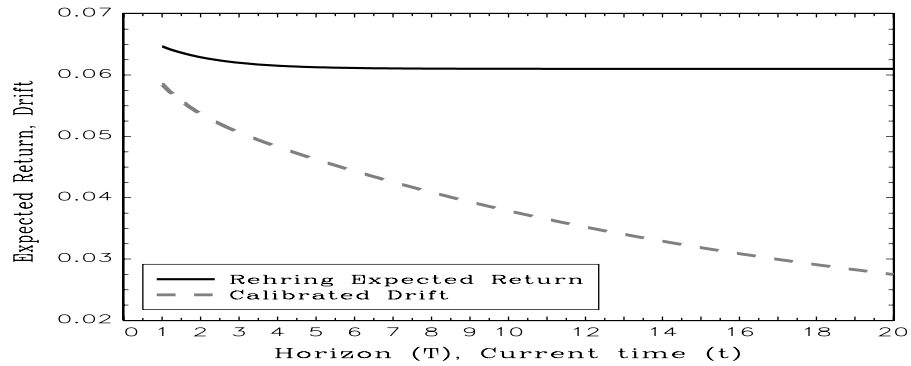


Figure 4: Calibration of the drift term structure of the stock

## 4.2 Standard deviation per year and volatility (term structure)

To fit the standard deviation per year, we can also use negative exponentials of type  $\alpha_\sigma - \beta_\sigma \exp(-\lambda_\mu t)$ . We get:

$$\begin{aligned}\alpha_\sigma^P &= 0.085; \beta_\sigma^P = -0.12; \lambda_\sigma^P = 0.45, \\ \alpha_\sigma^S &= 0.134; \beta_\sigma^S = -0.13; \lambda_\sigma^S = 0.455.\end{aligned}$$

Applying results of Appendix C, we determine the volatility of both the real estate asset and the stock, as shown in Figures 5 and 6.

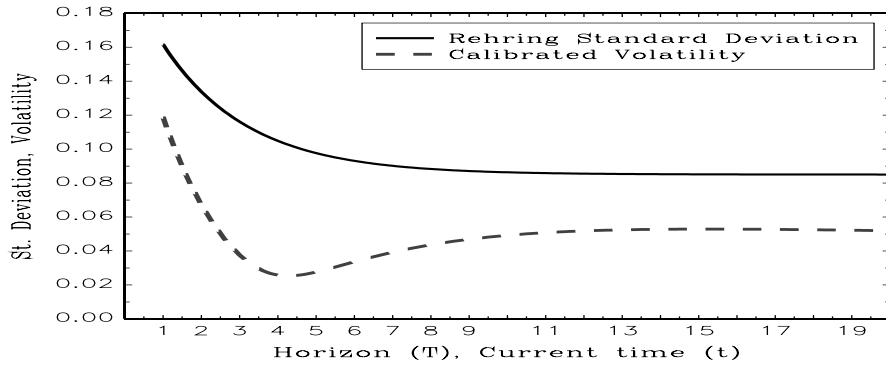


Figure 5: Calibration of the standard deviation per year  $T$  and the volatility term structure of the real estate asset, function of current time  $t$

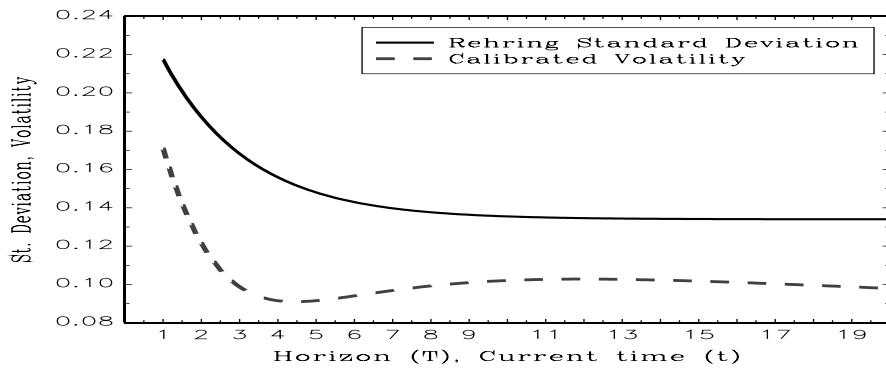


Figure 6: Calibration of the standard deviation per year  $T$  and the volatility term structure of the stock, function of current time  $t$

### 4.3 Correlation term structure

We search to calibrate the correlations of the three Brownian motions, namely the three parameters  $\rho^{r,P}$ ,  $\rho^{r,S}$  and  $\rho^{P,S}$ . Note that these correlations do not depend on current time  $t$  during a given management period  $[0, T]$ . But, for fixed horizon  $T$ , we can calibrate the constant parameters  $\rho_T^{r,P}$ ,  $\rho_T^{r,S}$  and  $\rho_T^{P,S}$  to actual data such as those of Rehring (2012), in particular from annualized correlations of cumulated asset returns (see Appendix C). Results are provided in Figures 7, 8 and 9.

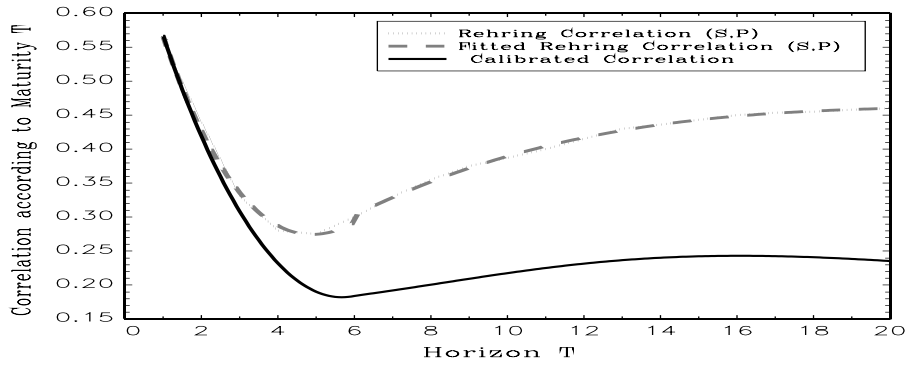


Figure 7: Calibration of the correlation of real estate and stock

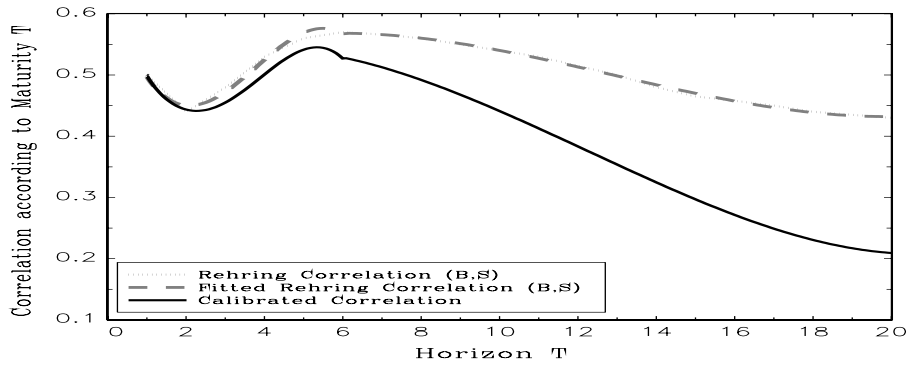


Figure 8: Calibration of the correlation of bond and stock

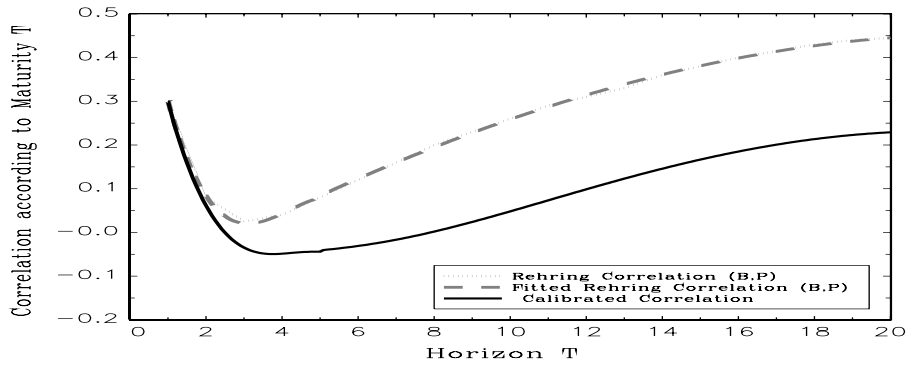


Figure 9: Calibration of correlation of bond and real estate

#### 4.4 Optimal weights (CRRA case)

For the numerical illustrations, we consider three types of investors with CRRA utilities: the first one is aggressive ( $\gamma = 3$ ), the second one is more moderate ( $\gamma = 5$ ), the third one is conservative ( $\gamma = 15$ ). We apply results of Proposition 4 to illustrate the optimal weights for the CRRA case. We begin by providing the optimal weights horizon, namely  $T = 20$  years. Figures 10, 11 and 12 show that the optimal weight on the real estate asset is always increasing with respect to current time, as in Rehring (2012) for  $t > 3$  years. At maturity ( $t = T = 20$ ), it reaches respectively about 115% for  $\gamma = 3$ , 60% for  $\gamma = 5$  and 20% for  $\gamma = 15$ . Both the weights on the bond and the stock are decreasing.

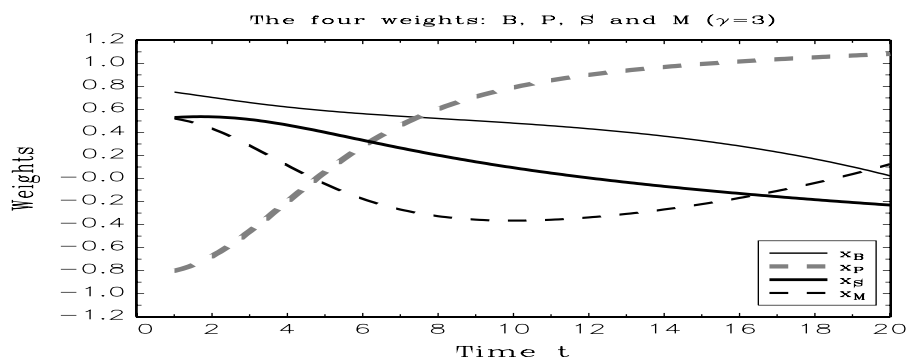


Figure 10: Weights for maturity of 20 years and relative risk aversion = 3

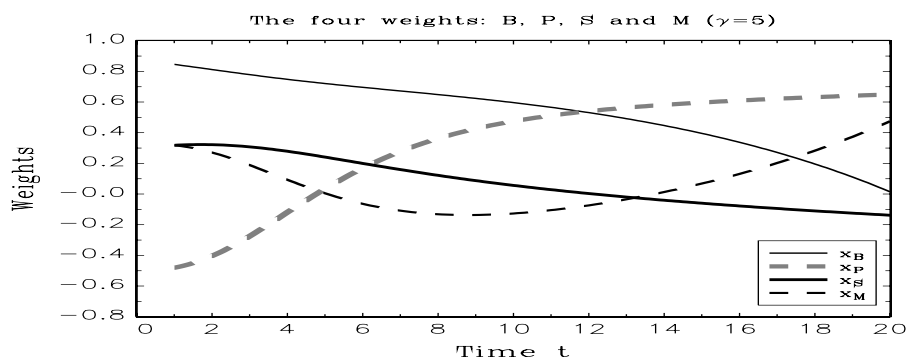


Figure 11: Weights for maturity of 20 years and relative risk aversion = 5

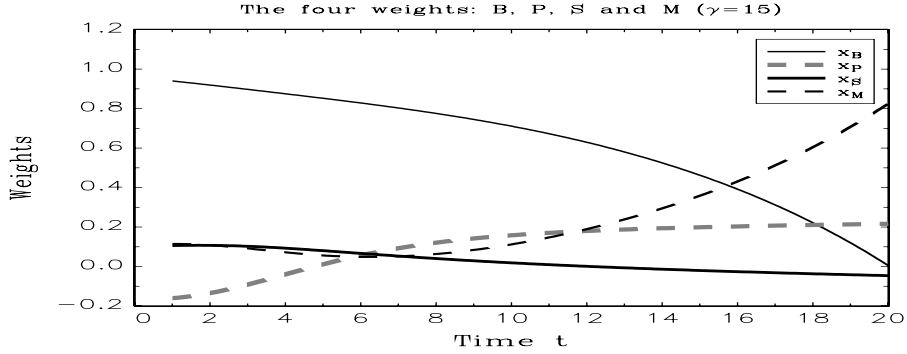


Figure 12: Weights for maturity of 20 years and relative risk aversion = 15

We examine now the optimal weights for a shorter horizon, namely  $T = 10$  years. Figures 13, 14 and 15 show that the optimal weight on the real estate asset is still increasing with respect to current time. At maturity ( $t = T = 10$ ), it reaches respectively about 75% for  $\gamma = 3$ , 45% for  $\gamma = 5$  and 10% for  $\gamma = 15$ . Both the weights on the bond and the stock are also still decreasing. Compared to the maturity of 20 years case, we note that, at a given current time  $t$ , weights at  $t = 10$  years are note equal to those for an horizon of 10 years. This is due to the term structure. For example, when dealing with a maturity of 20 years, the weights computed at  $t = 10$  years are respectively equal to 80% for  $\gamma = 3$ , 50% for  $\gamma = 5$  and 17% for  $\gamma = 15$ . Thus, they are higher than previous ones.

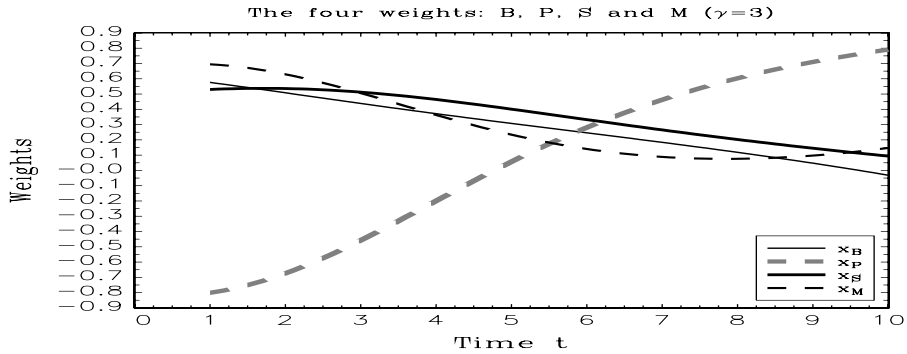


Figure 13: Weights for maturity of 10 years and relative risk aversion = 3

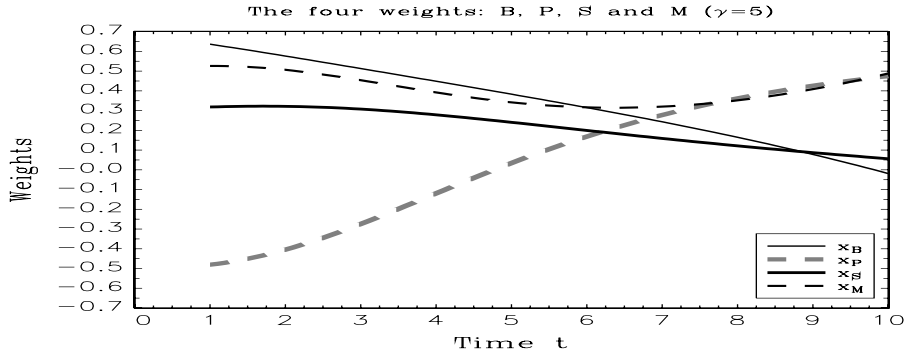


Figure 14: Weights for maturity of 10 years and relative risk aversion = 5

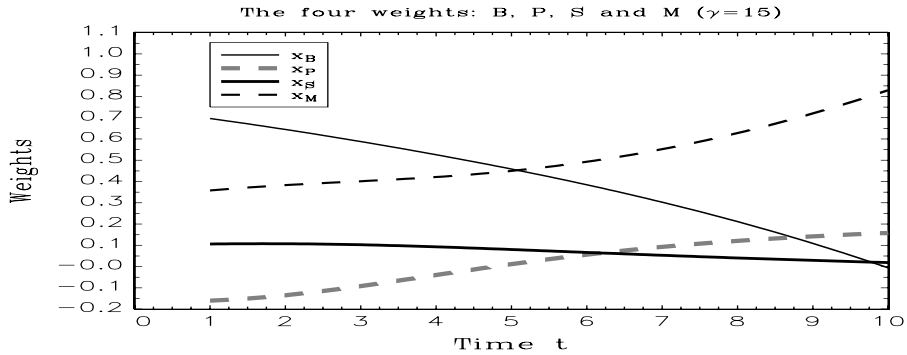


Figure 15: Weights for maturity of 10 years and relative risk aversion = 15

## 5 Conclusion

Real estate has traditionally been viewed as a low-risk asset class with good diversification properties. Given that most institutional investors have very long time horizons, typical portfolio optimizations based only on short-term returns, without accounting for return predictability, will produce results quite different from observed optimal allocation for these investors. The result of mean reversion is that the risk of real estate investment would be significantly reduced for long-term investors than for those with shorter investment horizons. This paper emphasizes the impact of the term structure of the asset returns, in particular the role of real estate market volatility on the optimal mix asset allocation. For this purpose, the investor is assumed to maximize the expected utility of her final wealth. We analyze in particular the optimal weights according to mean reverting assumptions about the term structure of the real estate asset. Our results show how optimal portfolio weights depend crucially on the investor's risk aversion and time horizon and also on the considered term structure of

asset returns. As emphasized by Feldman (2003), Fugazza *et al.* (2007) and Rehring (2012), the real estate allocation varies very significantly according to both risk aversion and portfolio horizon. We show also that it can be increasing with respect to current time and portfolio horizon. Finally, our findings allow to emphasize that, by considering mean reverting properties of asset returns, we get in particular real estate portfolio allocations closer to reality for long term investors having significant relative risk aversions, since the optimal weight on real estate asset lies between 10% and 20%.

A possible extension would be to examine the effects of market incompleteness<sup>12</sup> as for instance in Karatzas *et al.* (1991) and specific constraints on portfolio weights as in Cvitanic and Karatzas (1992).<sup>13</sup> In that case, it would be possible to determine the associated compensated variation, due to this lack of hedging against real asset risk, as computed for other market frictions in de Palma and Prigent (2009).

## References

- [1] Andonov, A., Kok, N. and Eichholtz, P. (2013). A global perspective on pension fund investments in real estate. *Journal of Portfolio Management*, 39, 32-42.
- [2] Bajeux-Besnainou, I., Jordan, J.V. and Portait, R. (2001). An asset allocation puzzle: comment. *American Economic Review*, 91, 1170-1179.
- [3] Barkham, R. J. and Geltner, D. M. (1995). Price discovery and efficiency in British and American property markets, *Real Estate Economics*, 23, 22-44.
- [4] Baroni, M., Barthélémy, F. and Mokrane, M. (2008). Un nouvel indice de risque immobilier pour le marché résidentiel parisien. *Revue Economique*, 59, 99-118.
- [5] Brennan, M.J., and Xia Y. (2000). Stochastic interest rates and the bond-stock mix. *European Finance Review*, 4, 197-210.
- [6] Brigo, D. and Mercurio, F. (2006). *Interest Rate Models - Theory and Practice*. Springer Finance, Springer-Verlag Berlin Heidelberg.
- [7] Campbell, J.Y. and Viceira, L.M., (2002). *Strategic asset allocation: Portfolio choice for long-term investors*. Oxford University Press: Oxford, UK and New York.

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<sup>12</sup>For example, if we assume that the real estate asset dynamics is driven by more than one Brownian motion if we consider multi factor modelling.

<sup>13</sup>Obviously, other models can be introduced and examined. For example, we can take account of stochastic market prices of risk, of labor income as in El Karoui and Jeanblanc (1998), for no traded asset such as in Lioui and Poncet (2001). Nevertheless note that, even in complete markets, Monte Carlo simulations are often necessary to compute optimal portfolios as for example in Detemple *et al.* (2003) or in Cvitanic *et al.* (2003).

- [8] Campbell, J.Y. and Viceira, L.M. (2005). The term structure of the risk-return trade-off. *Financial Analysts Journal*, 61, 34-44.
- [9] Canner, N., Mankiw, N.G., and Weil, D.N. (1997). An asset allocation puzzle. *American Economic Review*, 87, 181-191.
- [10] Chiarella, C., Semmler, W., Hsiao, C., and Mateane, L. (2016). Sustainable asset accumulation and dynamic portfolio decisions. *Dynamic Modeling and Econometrics in Economics and Finance* (volume 18) , Springer.
- [11] Chun, G. H., Sa-Aadu, J., and Shilling, J. D. (2004). The role of real estate in an institutional investor's portfolio revisited. *Journal of Real Estate Finance and Economics*, 29, 295-320.
- [12] Clayton, J. (2007). PREA Plan Sponsor Research Report. Pension Real Estate Association: Hartford, CT.
- [13] Cox, J. and Huang, C.F. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49, 33-83.
- [14] Cvitanic, J. and Karatzas, I. (1992). Convex duality in constrained portfolio optimization. *Annals of Applied Probability*, 2, 767-818.
- [15] Cvitanic, J., Goukasian, L. and Zapatero, F. (2003). Monte Carlo computation of optimal portfolios in complete markets. *J. Econ. Dyn. Control*, 27, 971-986.
- [16] de Palma, A. and Prigent, J.-L. (2009). Standardized versus customized portfolio: a compensating variation approach, *Annals of Operations Research*, 165, 161-185.
- [17] Detemple, J., Garcia, R. and Rindishbacher, M. (2003). A Monte Carlo method for optimal portfolios. *Journal of Finance*, 58, 401-446.
- [18] Duffie, D. and Kan, R. (1996). Yield-factor model of interest rates. *Mathematical Finance*, 6, 379-406.
- [19] El Karoui, N. and Jeanblanc, M. (1998). Optimization of consumption with labor income. *Finance and Stochastics*, 2, 409-440.
- [20] Fabozzi, F. (1996). *Bond Markets Analysis and Strategies*. 3rd ed. Englewood Cliffs, N.J: Prentice Hall.
- [21] Feldman, B. (2003). *Investment policy for securitized and direct real estate*. Chicago: Ibbotson Associates.
- [22] Fogler, H. (1984). 20% in real estate: Can theory justify it? *Journal of Portfolio Management*, 10, 6-13.

- [23] Fong, F. (1990). Portfolio construction: fixed income. In John L. Maginn and Donald L. Tuttle, eds., *Managing investment portfolios: a dynamic process*. 2nd ed. Boston: Warren, Gorham and Lamont.
- [24] Fugazza, C., Guidolin, M., and Nicodano, G. (2007). Investing for the long-run in European real estate. *Journal of Real Estate Finance and Economics*, 34, 35-80.
- [25] Geltner, D. (1991). Smoothing in appraisal-based returns, *Journal of Real Estate Finance and Economics*, 4, 327-345.
- [26] Geltner, D., J. Rodriguez and O'Connor, D. (1995). The similar genetics of public and private real estate and the optimal long-horizon portfolio mix. *Real Estate Finance*, 12, 71-81.
- [27] Gold, R. B. (1986). *Real estate: Can institutional portfolios be diversified without it?* Chicago: JMB Institutional Realty Corporation.
- [28] Gollier, C., (2001). *The Economics of Risk and Time*. The MIT Press, Cambridge, MA.
- [29] Heaney, R. and Srikanthakumar, S. (2012). Time-varying correlation between stock market returns and real estate returns. *Journal of Empirical Finance*, 19, 583-594.
- [30] Hoesli, M., Lekander, J. and Witkiewicz, W. (2004). New international evidence on real estate as a portfolio diversifier. *Journal of Real Estate Research*, 26, 161-206.
- [31] Hoesli, M., and MacGregor, B. D. (2000). *Property Investment*. Longman: Harlow, UK.
- [32] Hoesli, M., Oikarinen, E. and Serrano, C. (2015). Do public real estate returns really lead private returns? *Journal of Portfolio Management*. Special Real Estate Issue, 41, 105-117.
- [33] Hudson-Wilson, S., Gordon, J. N., Fabozzi, F. J., Anson, M. J. P. and Giliberto, S. M. (2005). Why Real Estate ? *Journal of Portfolio Management*, 31, 12-21.
- [34] Jacod, J. and Shiryaev, A. N. (2002). *Limit Theorems for Stochastic Processes* (2nd ed.). Springer-Verlag, Berlin- Heidelberg-New York.
- [35] J.P. Morgan. (2007). *The Alternative Asset Survey 2007*. JP Morgan Asset Management: Luxembourg.
- [36] Karatzas, I., Lehoczky, J.P., Shreve, S.E. and Xu, G.L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and optimization* 29, 702-730.

- [37] Lin, Z., and Liu, Y. (2008). Real estate returns and risk with heterogeneous investors. *Real Estate Economics*, 36, 753-776.
- [38] Lioui, A. and Poncet, P. (2001). On optimal portfolio choice under stochastic interest rates. *J. Econ. Dyn. Control*, 25, 1841-1865.
- [39] Long, J.B. (1990). The numeraire portfolio. *Journal of Financial Economics*, 26, 29-69.
- [40] Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, 7, 77-91.
- [41] MacKinnon, G.H. and Al Zaman, A., (2009). Real estate for the long term: The effect of return predictability on long-horizon allocations. *Real Estate Economics*, 37, 117-153.
- [42] Merton, R. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3, 373-413.
- [43] Pagliari, J. L. (2011). Long-run investment horizons and implications for mixed asset portfolio allocations. University of Chicago - Booth School of Business.
- [44] Pagliari, J. L. (2017). Another take on real estate's role in mixed-asset portfolio allocations. *Real Estate Economics*, 45, 75-132.
- [45] Prigent, J-L., (2007). Portfolio Optimization and Performance Analysis, Chapman & Hall, Boca Raton, Florida.
- [46] Rab, N. and Warnung, R. (2012). Scaling portfolio volatility and calculating risk contributions in the presence of serial cross-correlations, *Journal of Risk*, 14, 23-52.
- [47] Rehring, C. (2012). Real estate in a mixed-asset portfolio: The role of the investment horizon. *Real Estate Economics*, 40, 65-95.
- [48] Seiler, M.J., Webb, J.R. and Myer, F.C.N. (1999). Diversification issues in real estate investment. *Journal of Real Estate Literature*, 7, 163-179.
- [49] Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5, 177-88.
- [50] Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete Markets, *Journal of Financial and Quantitative Analysis*, 37, 63-91.
- [51] Wallace, N. E., and Meese, R. A. (1997). The construction of residential housing price indices: a comparison of repeat-sales, hedonic-regression, and hybrid approaches. *Journal of Real Estate Finance and Economics*, 14, 51-73.

- [52] Webb, J. R. and Rubens, J. H. (1986). Portfolio considerations in the valuation of real estate. *Real Estate Economics*, 14, 465-495.
- [53] Webb, J. R. and Rubens, J. H. (1987). How much in real estate? A surprising answer. *Journal of Portfolio Management*, 13, 10-14.

## Appendix

### Appendix A: Properties of the numeraire portfolio

Recall that the Radon-Nikodym density of the risk-neutral probability is given by:

$$\eta_t = \exp \left[ -\mathbf{M}_t - \int_0^t A(s) ds \right]$$

where

$$\mathbf{M}_t = \int_0^t \lambda_{r,s} dW_s^r + \int_0^t \lambda_{P,s} dW_s^P + \int_0^t \lambda_{S,s} dW_s^S,$$

and

$$A(t) = \frac{1}{2} (\lambda_{r,t}^2 + \lambda_{P,t}^2 + \lambda_{S,t}^2) + \lambda_{r,t} \lambda_{P,t} \rho^{r,P} + \lambda_{r,t} \lambda_{S,t} \rho^{r,S} + \lambda_{P,t} \lambda_{S,t} \rho^{P,S}.$$

The numeraire portfolio  $H$  is equal to:

$$H_t = \frac{\exp \left( \int_0^t r_s ds \right)}{\eta_t}.$$

Consequently, we get:

$$\ln H_t = \int_0^t r_s ds + \mathbf{M}_t + \int_0^t A(s) ds.$$

Since  $(W^r, W^P, W^S)$  is a Gaussian process, we can deduce that  $\ln H_t$  is Gaussian itself. We have to determine its expectation and variance. More precisely, we determine the conditional expectations for powers  $z$  of  $\mathbb{E}_t \left[ \frac{H_T^z}{H_t^z} \right]$ .

The ratio  $\frac{H_T^z}{H_t^z}$  satisfies:

$$\frac{H_T^z}{H_t^z} = \exp [N_z(t, T)],$$

with

$$N_z(t, T) = z \left[ \int_t^T r_s ds + \mathbf{M}_T - \mathbf{M}_t + \int_t^T A(s) ds \right].$$

The process  $N_z(t, T)$  is Gaussian distributed. Thus, to determine  $\mathbb{E}_t \left[ \frac{H_T^z}{H_t^z} \right]$ , we have just to compute  $\mathbb{E}_t [N_z(t, T)]$  and  $Var_t [N_z(t, T)]$ , since we have:

$$\mathbb{E}_t \left[ \frac{H_T^z}{H_t^z} \right] = \exp \left[ \mathbb{E}_t [N_z(t, T)] + \frac{1}{2} Var_t [N_z(t, T)] \right].$$

1) Determination of  $\mathbb{E}_t [N_z(t, T)]$ .

We have:

$$\mathbb{E}_t [N_z(t, T)] = z \left( \int_t^T \mathbb{E}_t [r_u] du + \int_t^T A(s) ds \right).$$

We deduce: for  $u > t$ ,

$$\mathbb{E}_t [r_u] = \bar{r} + (r_t - \bar{r}) e^{-(u-t)k_r}.$$

Finally, we get:

$$\mathbb{E}_t [N_z(t, T)] = z(T-t) \left( \bar{r} + (r_t - \bar{r}) \frac{\beta_r(T-t)}{T-t} + \frac{1}{T-t} \int_t^T A(s) ds \right),$$

which is equivalent to:

$$\mathbb{E}_t [N_z(t, T)] = z\Phi_{(t,T)},$$

with:

$$\Phi_{(t,T)} = (T-t) \left[ \bar{r} + (r_t - \bar{r}) \frac{\beta_r(T-t)}{(T-t)} + \frac{1}{T-t} \int_t^T A(s) ds \right].$$

2) Determination of  $Var_t [N_z(t, T)]$ .

Step 1: computation of  $Var_t \left[ \int_t^T r_s ds \right]$

- To calculate  $Var_t \left[ \int_t^T r_s ds \right]$ , we use the Fubini's property for stochastic integrals. We have:

$$r_t = \bar{r} + (r_s - \bar{r}) e^{-(t-s)k_r} + a_r e^{-(t-s)k_r} \int_s^t e^{(u-s)k_r} dW_u^r.$$

$$\begin{aligned} \int_t^T r_u du &= \bar{r}(T-t) + (r_t - \bar{r}) \beta_r(T-t) \\ &+ \int_t^T \left( a_r e^{-(u-t)k_r} \int_t^u e^{(v-t)k_r} dW_v^r \right) du. \end{aligned}$$

Then:

$$\begin{aligned} \int_t^T \left[ a_r e^{-(u-t)k_r} \int_t^u e^{(s-t)k_r} dW_s^r \right] du &= \int_t^T \left( \int_s^T a_r e^{-(u-t)k_r} e^{(s-t)k_r} du \right) dW_s^r, \\ &= \int_t^T \left( a_r \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] \right) dW_s^r, \end{aligned}$$

$$\begin{aligned} \text{Var}_t \left[ \int_t^T r_s ds \right] &= a_r^2 \int_t^T \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right]^2 ds, \\ \text{Var}_t \left[ \int_t^T r_s ds \right] &= \frac{a_r^2}{k_r^2} \left[ (T-t) + \frac{1}{2k_r} [1 - \exp(-2(T-t)k_r)] - 2\beta_r(T-t) \right]. \end{aligned}$$

Step 2: computation of  $\text{Var}_t [\mathbf{M}_T - \mathbf{M}_t]$ :

$$\text{Var}_t [\mathbf{M}_T - \mathbf{M}_t] = 2 \int_t^T A(s) ds.$$

Step 3: computation of  $\text{Cov}_t \left[ \int_t^T r_s ds; \mathbf{M}_T - \mathbf{M}_t \right]$ :

$$\text{Cov}_t \left[ \int_t^T r_s ds; \mathbf{M}_T - \mathbf{M}_t \right] = \int_t^T a_r (\lambda_{r,s} + \lambda_{P,s} \rho^{r,P} + \lambda_{S,s} \rho^{r,S}) \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds$$

To conclude, recall that:

$$\begin{aligned} \text{Var}_t [N_z(t, T)] &= \\ z^2 \left( \text{Var}_t \left[ \int_t^T r_s ds \right] + \text{Var}_t [\mathbf{M}_T - \mathbf{M}_t] + 2 \text{Cov}_t \left[ \int_t^T r_s ds, \mathbf{M}_T - \mathbf{M}_t \right] \right). \end{aligned}$$

Therefore, we get:

$$\begin{aligned} \text{Var}_t [N_z(t, T)] &= \\ z^2 \left( \begin{aligned} &\frac{a_r^2}{k_r^2} \left[ (T-t) + \frac{1}{2k_r} [1 - \exp(-2(T-t)k_r)] - 2\beta_r(T-t) \right] \\ &+ 2 \int_t^T A(s) ds + 2 \int_t^T a_r (\lambda_{r,s} + \lambda_{P,s} \rho^{r,P} + \lambda_{S,s} \rho^{r,S}) \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds \end{aligned} \right). \end{aligned}$$

## Appendix B: Optimal weights

### CARA case

At time  $t$ , the optimal CARA portfolio value is given by:

$$V_t^{exp} = \left[ A(V_0) \tilde{\alpha}_{N,(t,T)} + \frac{1}{a} \tilde{\beta}_{N,(t,T)} \right] + \frac{1}{a} \ln(H_t/I_t) \tilde{\alpha}_{N,(t,T)}. \quad (34)$$

To compute the optimal weights  $x_B$ ,  $x_P$  and  $x_S$ , first we calculate  $\frac{dV_t^{exp}}{V_t^{exp}}$  by using Relation (34). Note that  $\tilde{\alpha}_N$  depends only on  $r_t$ , while  $\tilde{\beta}_N$  is a deterministic function. Thus, applying Ito's formula, we obtain:

$$\begin{aligned} dV_t^{exp} &= \tilde{A}(V_0) \left[ \frac{\partial \tilde{\alpha}_N}{\partial t} dt + \frac{\partial \tilde{\alpha}_N}{\partial r} dr_t \right] + \tilde{A}(V_0) \left[ \frac{1}{2} a_r^2 \frac{\partial^2 \tilde{\alpha}_N}{\partial r^2} \right] dt \\ &\quad + \frac{1}{a} \left( \left[ \frac{\partial \tilde{\beta}_N}{\partial t} dt + \frac{\partial \tilde{\beta}_N}{\partial r} dr_t \right] + \frac{1}{2} \left[ a_r^2 \frac{\partial^2 \tilde{\beta}_N}{\partial r^2} \right] dt \right) \\ &\quad + \frac{\ln(H_t)}{a} \left( \left[ \frac{\partial \tilde{\alpha}_N}{\partial t} dt + \frac{\partial \tilde{\alpha}_N}{\partial r} dr_t \right] + \frac{1}{2} \left[ a_r^2 \frac{\partial^2 \tilde{\alpha}_N}{\partial r^2} \right] dt \right) \\ &\quad + \frac{1}{a} \tilde{\alpha}_N \left[ \frac{dH_t}{H_t} - \frac{1}{2H_t^2} d\langle H, H \rangle_t \right] \\ &\quad - \frac{1}{a} \tilde{\alpha}_N (T-t) [a_r \beta_r (\lambda_{r,t} + \lambda_{P,t} \rho^{r,P} + \lambda_{S,t} \rho^{r,S})] dt. \end{aligned} \quad (35)$$

where  $\langle H, H \rangle_t$  is the instantaneous variances of  $H_t$ . Define  $\chi_{r,t}$  by:

$$\chi_{r,t} = \left( \frac{1}{a} \right) \left[ aA(V_0) \frac{\partial \tilde{\alpha}_N}{\partial r} + \frac{\partial \tilde{\beta}_N}{\partial r} + \ln(H_t) \frac{\partial \tilde{\alpha}_N}{\partial r} \right].$$

Using the definitions of the functions  $\tilde{\alpha}_N$  and  $\tilde{\beta}_N$ , we obtain:

$$\chi_{r,t} = A(V_0) \tilde{\alpha}_N (-(T-t)\beta_r) + \left( \frac{1}{a} \right) \times$$

$$\left[ \tilde{\alpha}_N ((T-t)\beta_r) (\Psi_{(t,T)} - \Phi_{(t,T)}) + \tilde{\alpha}_N ((T-t)\beta_r) \right] + \ln(H_t) \tilde{\alpha}_N (-(T-t)\beta_r),$$

and finally:

$$\chi_{r,t} = \left( \frac{1}{a} \right) \tilde{\alpha}_N (T-t)\beta_r \left[ -a\tilde{A}(V_0) + [\Psi_{(t,T)} - \Phi_{(t,T)} + 1] - \ln(H_t) \right].$$

Then, by identifying the Brownian integrals of Relations (19) and (35), we obtain the following system:

$$\begin{cases} \frac{\chi_{r,t} a_r a + \alpha_N \lambda_{r,t}}{a V_t^{exp}} = -x_B(t) \beta_r (D_B) a_r \\ \frac{\alpha_N \lambda_{P,t}}{a V_t^{exp}} = x_P(t) \sigma_t^P \\ \frac{\alpha_N \lambda_{S,t}}{a V_t^{exp}} = x_S(t) \sigma_t^S \end{cases}$$

Therefore, we deduce that, at any time  $t$ , the weights are given by:

$$\begin{cases} x_B^{CARA}(t) &= -\frac{1}{\beta_r(D_B)a_r} \left( \frac{\chi_{r,t}a_r a + \alpha_N \lambda_{r,t}}{aV_t^{exp}} \right) \\ x_P^{CARA}(t) &= \frac{1}{\sigma_t^P} \left( \frac{\alpha_N \lambda_{P,t}}{aV_t^{exp}} \right) \\ x_S^{CARA}(t) &= \frac{1}{\sigma_t^S} \left( \frac{\alpha_N \lambda_{S,t}}{aV_t^{exp}} \right) \\ x_C^{CARA}(t) &= 1 - x_B^{CARA} - x_P^{CARA} - x_S^{CARA} \end{cases}$$

Note that these optimal weights are deterministic functions of the instantaneous interest rate  $r_t$  and both the numeraire portfolio  $H_t$  and the portfolio  $V_t^{exp}$  itself. These functions depend only on market parameters.

### CRRA case

We have:

$$V_T^{CRRA} = \left( \frac{V_0}{\mathbb{E}[H_T^{(\frac{1-\gamma}{\gamma})}]} \right) H_T^{(\frac{1}{\gamma})}. \quad (36)$$

Denote

$$\tilde{A} = \left( \frac{V_0}{\mathbb{E}[H_T^{(\frac{1-\gamma}{\gamma})}]} \right).$$

To compute the optimal weights  $x_C, x_B, x_P$ , and  $x_S$ , first we calculate  $V_t^{CRRA}$ . We have:

$$V_t^{CRRA} \frac{1}{H_t} = \mathbb{E}_t \left[ \frac{1}{H_T} V_T^{CRRA} \right] = \tilde{A} H_t^{(\frac{1}{\gamma})} \mathbb{E}_t \left[ (H_T/H_t)^{(\frac{1}{\gamma}-1)} \right].$$

which implies

$$V_t^{CRRA} = \tilde{A} H_t^{(\frac{1}{\gamma})} \mathbb{E}_t [(H_T/H_t)^z] \text{ with } z = \frac{1}{\gamma} - 1.$$

Using Relation (25), we get:

$$V_t^{CRRA} = \tilde{A} H_t^{(\frac{1}{\gamma})} \exp \left[ z \Phi_{(t,T)} + \frac{1}{2} z^2 \Psi_{(0,T)} \right] \text{ with } z = \frac{1}{\gamma} - 1.$$

Then, we determine  $\frac{dV_t^{CRRA}}{V_t^{CRRA}}$  by using Relation (36). Applying Ito's formula, we get the martingale part of  $\frac{dV_t^{CRRA}}{V_t^{CRRA}}$ :

$$\begin{aligned} dV_t^{CRRA} &= (\dots)dt \\ &+ \frac{1}{\gamma} V_t^{CRRA} (\lambda_{r,t} dW_t^r + \lambda_{P,t} dW_t^P + \lambda_{S,t} dW_t^S) \\ &+ V_t^{CRRA} z [\beta_r a_r dW_t^r] \end{aligned}$$

We obtain the following system:

$$\begin{cases} \frac{\lambda_{r,t}}{\gamma} + za_r\beta_r(T-t) & = \delta_r(t) = -x_B(t)\beta_r(D_B)a_r \\ \frac{\lambda_{P,t}}{\gamma} & = \delta_P(t) = x_P(t)\sigma_t^P \\ \frac{\lambda_{S,t}}{\gamma} & = \delta_S(t) = x_S(t)\sigma_t^S \end{cases}$$

Therefore, we deduce that, at any time  $t$ , the weights are given by:

$$\begin{cases} x_B^{CRR A}(t) & = -\frac{\delta_i(t)}{\beta_i(D)a_i} \\ x_P^{CRR A}(t) & = \frac{\delta_P(t)}{\sigma_t^P} \\ x_S^{CRR A}(t) & = \frac{\delta_S(t)}{\sigma_t^S} \\ x_C^{CRR A}(t) & = 1 - x_B^{CRR A} - x_P^{CRR A} - x_S^{CRR A} \end{cases}$$

Recall that  $\Sigma$  is the matrix:

$$\Sigma = \begin{bmatrix} -\beta_r(D_B)a_r & 0 & 0 \\ 0 & \sigma_t^P & 0 \\ 0 & 0 & \sigma_t^S \end{bmatrix}.$$

We have:

$$\begin{bmatrix} \frac{\lambda_{r,t}}{\gamma} + za_r\beta_r(T-t) \\ \frac{\lambda_P}{\gamma} \\ \frac{\lambda_S}{\gamma} \end{bmatrix} = {}^t\Sigma \cdot \begin{bmatrix} x_B^{CRR A}(t) \\ x_P^{CRR A}(t) \\ x_S^{CRR A}(t) \end{bmatrix},$$

thus:

$$\begin{bmatrix} x_B^{CRR A}(t) \\ x_P^{CRR A}(t) \\ x_S^{CRR A}(t) \end{bmatrix} = {}^t\Sigma^{-1} \cdot \left( \begin{bmatrix} \frac{\lambda_{r,t}}{\gamma} + za_r\beta_r(T-t) \\ \frac{\lambda_{P,t}}{\gamma} \\ \frac{\lambda_{S,t}}{\gamma} \end{bmatrix} \right).$$

Recall that  $z = \frac{1}{\gamma} - 1$  and that the optimal portfolio for the logarithm case  $X^{Log}$  is given by:

$$\begin{bmatrix} x_B^{Log}(t) \\ x_P^{Log}(t) \\ x_S^{Log}(t) \end{bmatrix} = {}^t\Sigma^{-1} \cdot \left( \begin{bmatrix} \lambda_{r,t} \\ \lambda_{P,t} \\ \lambda_{S,t} \end{bmatrix} \right).$$

Therefore, we have:

$$X^{CRR A} = \frac{1}{\gamma}X^{Log} + \left(1 - \frac{1}{\gamma}\right)X^{Conservative},$$

where  $X^{Log}$  is the optimal portfolio for the logarithm case, and  $X^{Conservative}$  corresponds to the optimal portfolio for an infinite relative risk aversion  $\gamma$ . This latter portfolio is given by:

$$\begin{bmatrix} x_B^{Conservative}(t) \\ x_P^{Conservative}(t) \\ x_S^{Conservative}(t) \end{bmatrix} = {}^t\Sigma^{-1} \cdot \left( \begin{bmatrix} -a_r\beta_r(T-t) \\ 0 \\ 0 \end{bmatrix} \right).$$

## Appendix C (Calibration)

In what follows, we detail how we can calibrate drifts, volatilities and instantaneous correlations of the Brownian motions to respectively expected returns per year, standard deviations per year and assets correlations. Indeed, most of the empirical results about mean reverting asset returns are based on cumulated returns on given time periods  $[0, T]$  which are further annualized (see e.g. Campbell and Viceira, 2002, 2005; Rehring, 2012; Pagliari, 2017). Using a continuous-time approach, we have to identify the parameters corresponding to instantaneous variations using those which correspond to annualized characteristics of cumulated returns.

**Notation 7** We use the following notations (see Jacod and Shiryaev, 2002). For any two semimartingales  $X$  and  $Y$ ,  $[X, Y]$  denotes the quadratic variation of the processes  $X$  and  $Y$ . The process  $\langle X, Y \rangle$  denotes the predictable compensator of these processes. Recall that we have:

$$[X, Y]_t = \langle X, Y \rangle_t + \sum_{s \leq t} \Delta X_s \Delta Y_s,$$

where  $\Delta X_t$  and  $\Delta Y_t$  denote the jumps of the processes and  $\langle X, Y \rangle_t$  is defined from the following condition: The respective martingales parts  $M^X$  and  $M^Y$  of  $X$  and  $Y$  are such that  $(M_t^X M_t^Y - \langle X, Y \rangle_t)_t$  is a (local) martingale.

We have: (integration by part formula)

$$d(XY) = XdY + YdX + d[X, Y].$$

The process  $\mathcal{E}(Z)$  denotes the Dade-Doléans stochastic exponential, defined from the stochastic differential equation (SDE):

$$d\mathcal{E}(Z) = \mathcal{E}(Z)dZ.$$

In what follows, we consider two financial assets  $X$  and  $Y$  defined by:

$$dX_t = X_t(\mu_t^X dt + \sigma_t^X dW_t^X), \quad (37)$$

$$dY_t = Y_t(\mu_t^Y dt + \sigma_t^Y dW_t^Y), \quad (38)$$

where  $W_t = (W_t^X, W_t^Y)_{1 \leq i \leq d}$  is a 2-dimensional Brownian motion with correlation matrix  $\Sigma_{c,T}$  given by

$$\Sigma_{c,T} = \begin{bmatrix} 1 & \rho_T^{W^X, W^Y} \\ \rho_T^{W^X, W^Y} & 1 \end{bmatrix},$$

where  $\rho_T^{W^X, W^Y} = \langle W_t^X, W_t^Y \rangle_t$  is a constant calibrated for a fixed management period  $[0, T]$ .

Denote:

$$\Sigma_t = \begin{bmatrix} \sigma_t^X & 0 \\ 0 & \sigma_t^Y \end{bmatrix}.$$

### Computation of expected returns and variances

The random variable  $X_T$  can be expressed as  $X_T = \exp [N_T^X]$  where  $N_T^X$  has a Gaussian distribution with

$$\mathbb{E} [N_T^X] = \Phi_T^X,$$

where

$$\Phi_T^X = \left[ \bar{r}T + (r_0 - \bar{r})\beta_r(T) + \int_0^T \left( \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds \right],$$

and

$$\text{Var} [N_T^X] = \Psi_T^X$$

with

$$\begin{aligned} \Psi_T^X &= \frac{a_r^2}{k_r^2} \left[ T + \frac{1}{2k_r} [1 - \exp(-2Tk_r)] - 2\beta_r(T) \right] \\ &+ \int_0^T [\sigma_s^X]^2 ds + 2a_r \rho_T^{Wr, W^X} \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(t-s)k_r}}{k_r} \right] ds \end{aligned}$$

Therefore, we have:

$$\mathbb{E} [X_T] = X_0 \exp \left[ \Phi_T^X + \frac{1}{2} \Psi_T^X \right],$$

and the variance is given by:

$$\text{Variance} [X_T] = X_0^2 \exp [2\Phi_T^X + \Psi_T^X] (\exp [\Psi_T^X] - 1).$$

**Proof.** We have:

$$dX_t = X_t [(r_t + \theta_t^X) dt + \sigma_t^X dW_t^X],$$

which implies that:

$$X_T = X_0 \exp \left[ \int_0^T \left( r_s + \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds + \int_0^T \sigma_s^X dW_s^X \right].$$

Since  $\int_0^T r_s ds$  and  $\int_0^T \sigma_s^X dW_s^X$  have Gaussian distributions,  $X_T$  has a Log-normal distribution. We use similar properties as in Appendix A.

Denote:

$$N_T^X = \int_0^T \left( r_s + \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds + \int_0^T \sigma_s^X dW_s^X.$$

We have:

$$\mathbb{E} [N_T^X] = \int_0^T \left( \mathbb{E} [r_s] + \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds$$

$$\mathbb{E} [N_T^X] = \left( \bar{r}T + (r_0 - \bar{r})\beta_r(T) + \int_0^T \left( \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds \right),$$

which is equivalent to:

$$\mathbb{E} [N_T^X] = \Phi_T^X,$$

with:

$$\Phi_T^X = \left[ \bar{r}T + (r_0 - \bar{r})\beta_r(T) + \int_0^T \left( \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds \right].$$

2) Determination of  $Var [N_T^X]$ .

Step 1: computation of  $Var \left[ \int_0^T r_s ds \right]$

$$Var \left[ \int_0^T r_s ds \right] = \frac{a_r^2}{k_r^2} \left[ T + \frac{1}{2k_r} [1 - \exp(-2Tk_r)] - 2\beta_r(T) \right].$$

Step 2: computation of  $Var \left[ \int_0^T \sigma_s^X dW_s^X \right]$ :

$$Var \left[ \int_0^T \sigma_s^X dW_s^X \right] = \int_0^T [\sigma_s^X]^2 ds.$$

Step 3: computation of  $Cov \left[ \int_0^T r_s ds; \int_0^T \sigma_s^X dW_s^X \right]$ :

$$Cov \left[ \int_0^T r_s ds; \int_0^T \sigma_s^X dW_s^X \right] = \int_0^T a_r \rho_T^{W_r, W^X} \sigma_s^X \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds$$

To conclude, recall that:

$$Var [N_T^X] = Var \left[ \int_0^T r_s ds \right] + Var \left[ \int_0^T \sigma_s^X dW_s^X \right] + 2Cov \left[ \int_0^T r_s ds; \int_0^T \sigma_s^X dW_s^X \right].$$

Therefore, we get: (notation:  $Var [N_T^X] = \Psi_T^X$ )

$$Var [N_T^X] = \Psi_T^X = \frac{a_r^2}{k_r^2} \left[ T + \frac{1}{2k_r} [1 - \exp(-2Tk_r)] - 2\beta_r(T) \right] + \int_0^T [\sigma_s^X]^2 ds + 2a_r \rho_T^{W_r, W^X} \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(t-s)k_r}}{k_r} \right] ds.$$

Finally, we get:

$$\mathbb{E} [X_T] = X_0 \exp \left[ \Phi_T^X + \frac{1}{2} \Psi_T^X \right].$$

Using the Lognormal distribution properties, we deduce:

$$Variance [X_T] = X_0^2 \exp [2\Phi_T^X + \Psi_T^X] (\exp [\Psi_T^X] - 1).$$

■

**Calibration to the given return per year  $g(T)$  and to the given standard deviation per year  $h(T)$ :**

We must have:

$$\begin{aligned}\frac{1}{T} \exp \left[ \Phi_T^X + \frac{1}{2} \Psi_T^X \right] &= g(T), \\ \frac{1}{T} \exp \left[ 2\Phi_T^X + \Psi_T^X \right] (\exp [\Psi_T^X] - 1) &= h(T).\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\exp \left[ \Phi_T^X + \frac{1}{2} \Psi_T^X \right] &= Tg(T), \\ \exp [\Psi_T^X] &= 1 + \frac{h(T)}{Tg^2(T)}.\end{aligned}$$

which is equivalent to:

$$\begin{aligned}\exp [\Phi_T^X] &= Tg(T) / \sqrt{1 + \frac{h(T)}{Tg^2(T)}}, \\ \exp [\Psi_T^X] &= 1 + \frac{h(T)}{Tg^2(T)}.\end{aligned}$$

and also:

$$\begin{aligned}\Phi_T^X &= \text{Log} \left[ Tg(T) / \sqrt{1 + \frac{h(T)}{Tg^2(T)}} \right], \\ \Psi_T^X &= \text{Log} \left[ 1 + \frac{h(T)}{Tg^2(T)} \right].\end{aligned}$$

Therefore, we deduce:

$$\begin{aligned}\text{For all } T, \frac{\partial \Phi_T^X}{\partial T} &= \frac{\partial}{\partial T} \text{Log} \left[ Tg(T) / \sqrt{1 + \frac{h(T)}{Tg^2(T)}} \right], \\ \frac{\partial \Psi_T^X}{\partial T} &= \frac{\partial}{\partial T} \text{Log} \left[ 1 + \frac{h(T)}{Tg^2(T)} \right]. \\ \\ \frac{\partial \Phi_T^X}{\partial T} &= \frac{\partial}{\partial T} [\bar{r}T + (r_0 - \bar{r})\beta_r(T)] + \left( \theta_T^X - \frac{1}{2} [\sigma_T^X]^2 \right) \\ &= \frac{\partial}{\partial T} \text{Log} \left[ Tg(T) / \sqrt{1 + \frac{h(T)}{Tg^2(T)}} \right]\end{aligned}$$

Consequently, we have:

$$\left( \theta_T^X - \frac{1}{2} [\sigma_T^X]^2 \right) =$$

$$\frac{\partial}{\partial T} \text{Log} \left[ Tg(T) / \sqrt{1 + \frac{h(T)}{Tg^2(T)}} \right] - \frac{\partial}{\partial T} [\bar{r}T + (r_0 - \bar{r})\beta_r(T)]$$

To get the standard deviation, we note that:

$$\begin{aligned} \frac{\partial \Psi_T^X}{\partial T} &= \frac{\partial}{\partial T} \left( \frac{a_r^2}{k_r^2} \left[ T + \frac{1}{2k_r} [1 - \exp(-2Tk_r)] - 2\beta_r(T) \right] \right) \\ &\quad + [\sigma_T^X]^2 + 2a_r \frac{\partial}{\partial T} \left( \rho_T^{W_r, W^X} \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(t-s)k_r}}{k_r} \right] ds \right) \\ &= \frac{\partial}{\partial T} \text{Log} \left[ 1 + \frac{h(T)}{Tg^2(T)} \right] \end{aligned}$$

Finally, we get:

$$\begin{aligned} &[\sigma_T^X]^2 + 2a_r \frac{\partial}{\partial T} \left( \rho_T^{W_r, W^X} \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds \right) \\ &= \frac{\partial}{\partial T} \text{Log} \left[ 1 + \frac{h(T)}{Tg^2(T)} \right] - \frac{\partial}{\partial T} \left( \frac{a_r^2}{k_r^2} \left[ T + \frac{1}{2k_r} [1 - \exp(-2Tk_r)] - 2\beta_r(T) \right] \right), \end{aligned}$$

where we have:

$$\begin{aligned} &\frac{\partial}{\partial T} \left( \rho_T^{W_r, W^X} \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds \right) = \\ &\frac{\partial}{\partial T} \left( \rho_T^{W_r, W^X} \right) \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds + \rho_T^{W_r, W^X} \frac{\partial}{\partial T} \left( \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds \right). \end{aligned}$$

### Computation of covariances and correlations

In what follows, we search to calibrate the correlations of the three Brownian motions, namely the three parameters  $\rho_T^{r, P}$ ,  $\rho_T^{r, S}$  and  $\rho_T^{P, S}$  to Rehring (2012) data (annualized correlations of cumulated asset returns).

For this purpose, let us denote:

$$\begin{aligned} A_t &= E \left[ \int_0^t r_s ds \right] = \bar{r}t + (r_0 - \bar{r})\beta_r(t); \\ B_t &= \frac{a_r^2}{k_r^2} \left[ t + \frac{1}{2k_r} [1 - \exp(-2tk_r)] - 2\beta_r(t) \right]; \\ C_t &= a_r \rho_T^{W_r, W^X} \int_0^t \sigma_s^X \left[ \frac{1 - e^{-(t-s)k_r}}{k_r} \right] ds; \\ D_t &= a_r \rho_T^{W_r, W^Y} \int_0^t \sigma_s^Y \left[ \frac{1 - e^{-(t-s)k_r}}{k_r} \right] ds. \end{aligned}$$

Taking account of the interest rate randomness, the covariance of asset prices  $X$  and  $Y$  is given by:

$$\text{Cov}_T^{X, Y} =$$

$$X_0 Y_0 \exp \left[ \int_0^T (E[r_s] + \theta_s^X) ds + \int_0^T (E[r_s] + \theta_s^Y) ds + (B_T + C_T + D_T) \right] \\ \left( \exp \left[ \int_0^T \sigma_s^X \sigma_s^Y \rho_T^{W^X, W^Y} ds + (B_T + C_T + D_T) \right] - 1 \right).$$

**Proof.** We have:

$$\begin{aligned} X_t &= X_0 \mathcal{E} \left[ \mu_s^X ds + \int_0^T \sigma_s^X dW_s^X \right] \\ &= \exp \left[ \int_0^T \left( \mu_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds + \int_0^T \sigma_s^X dW_s^X \right], \\ Y_t &= Y_0 \mathcal{E} \left[ \mu_s^Y ds + \int_0^T \sigma_s^Y dW_s^Y \right] \\ &= Y_0 \exp \left[ \int_0^T \left( \mu_s^Y - \frac{1}{2} [\sigma_s^Y]^2 \right) ds + \int_0^T \sigma_s^Y dW_s^Y \right], \end{aligned}$$

from which we get by using Yor's formula:

$$\begin{aligned} X_t Y_t &= X_0 Y_0 \mathcal{E} \left[ \mu_s^X ds + \int_0^T \sigma_s^X dW_s^X \right] \mathcal{E} \left[ \mu_s^Y ds + \int_0^T \sigma_s^Y dW_s^Y \right] = \\ X_0 Y_0 \exp &\left[ \int_0^T \left( \mu_s^X - \frac{1}{2} [\sigma_s^X]^2 + \mu_s^Y - \frac{1}{2} [\sigma_s^Y]^2 \right) ds \right. \\ &\left. + \int_0^T \sigma_s^X dW_s^X + \int_0^T \sigma_s^Y dW_s^Y + \left[ \int_0^T \sigma_s^X dW_s^X, \int_0^T \sigma_s^Y dW_s^Y \right] \right] = \\ X_0 Y_0 \exp &\left[ \int_0^T \left( \mu_s^X - \frac{1}{2} [\sigma_s^X]^2 + \mu_s^Y - \frac{1}{2} [\sigma_s^Y]^2 + \sigma_s^X \sigma_s^Y \rho_T^{W^X, W^Y} \right) ds \right. \\ &\left. + \int_0^T \sigma_s^X dW_s^X + \int_0^T \sigma_s^Y dW_s^Y \right] \end{aligned}$$

Recall that:  $\mu_s^X = r_s + \theta_s^X$  and  $\mu_s^Y = r_s + \theta_s^Y$ , and:

$$Cov_T^{X,Y} = \mathbb{E}[X_T Y_T] - \mathbb{E}[X_T] \mathbb{E}[Y_T]$$

Step 1: Computation of  $\mathbb{E}[X_T Y_T]$

We have:

$$\begin{aligned} X_T Y_T &= X_0 Y_0 \times \\ \exp &\left[ \int_0^T \left( r_s + \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 + r_s + \theta_s^Y - \frac{1}{2} [\sigma_s^Y]^2 + \sigma_s^X \sigma_s^Y \rho_T^{W^X, W^Y} \right) ds \right. \\ &\left. + \int_0^T \sigma_s^X dW_s^X + \int_0^T \sigma_s^Y dW_s^Y \right]. \end{aligned}$$

Denote by  $N_T^{X,Y}$  the random variable such that  $X_T Y_T = \exp [N_T^{X,Y}]$ . The random variable  $N_T^{X,Y}$  has a Gaussian distribution with:

$$\mathbb{E} [N_T^{X,Y}] = \Phi_T^{X,Y},$$

where

$$\Phi_T^{X,Y} = \left[ 2(\bar{r}T + (r_0 - \bar{r})\beta_r(T)) + \int_0^T \left[ \left( \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) + \left( \theta_s^Y - \frac{1}{2} [\sigma_s^Y]^2 \right) \right] ds \right].$$

Note that we have:

$$\begin{aligned} \Phi_T^X &= A_T + \int_0^T \left( \theta_s^X - \frac{1}{2} [\sigma_s^X]^2 \right) ds \\ \Phi_T^Y &= A_T + \int_0^T \left( \theta_s^Y - \frac{1}{2} [\sigma_s^Y]^2 \right) ds \\ \Phi_T^{X,Y} &= \Phi_T^X + \Phi_T^Y \end{aligned}$$

The random variable  $N_T^{X,Y}$  has a Gaussian distribution with:

$$\text{Var} \left[ N_T^{X,Y} \right] = \Psi_T^{X,Y}.$$

with

$$\begin{aligned} \Psi_T^{X,Y} &= 4 \frac{a_r^2}{k_r^2} \left[ T + \frac{1}{2k_r} [1 - \exp(-2Tk_r)] - 2\beta_r(T) \right] + \\ &\int_0^T [\sigma_s^X]^2 ds + \int_0^T [\sigma_s^Y]^2 ds + 2\rho_T^{W^X, W^Y} \int_0^T \sigma_s^X \sigma_s^Y ds + \\ 4a_r \rho_T^{W^r, W^X} \int_0^T \sigma_s^X \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds + 4a_r \rho_T^{W^r, W^Y} \int_0^T \sigma_s^Y \left[ \frac{1 - e^{-(T-s)k_r}}{k_r} \right] ds. \end{aligned}$$

Note that we have:

$$\begin{aligned} \Psi_T^X &= B_T + \int_0^T [\sigma_s^X]^2 ds + 2C_T; \\ \Psi_T^Y &= B_T + \int_0^T [\sigma_s^Y]^2 ds + 2D_T; \\ \Psi_T^{X,Y} &= \Psi_T^X + \Psi_T^Y + 2 \int_0^T \sigma_s^X \sigma_s^Y \rho_T^{W^X, W^Y} ds + 2(B_T + C_T + D_T). \end{aligned}$$

Therefore, since  $N_T^{X,Y}$  has a Gaussian distribution, we get:

$$\mathbb{E}[X_T Y_T] = X_0 Y_0 \exp \left[ \Phi_T^{X,Y} + \frac{1}{2} \Psi_T^{X,Y} \right].$$

Additionally, we have:

$$\mathbb{E}[X_T] \mathbb{E}[Y_T] = X_0 Y_0 \exp \left[ (\Phi_T^X + \Phi_T^Y) + \frac{1}{2} (\Psi_T^X + \Psi_T^Y) \right].$$

Consequently, we get:

$$Cov_T^{X,Y} = X_0 Y_0 \left( \exp \left[ \Phi_T^{X,Y} + \frac{1}{2} \Psi_T^{X,Y} \right] - \exp \left[ (\Phi_T^X + \Phi_T^Y) + \frac{1}{2} (\Psi_T^X + \Psi_T^Y) \right] \right).$$

Thus

$$Cov_T^{X,Y} = X_0 Y_0 \exp \left[ (\Phi_T^X + \Phi_T^Y) + \frac{1}{2} (\Psi_T^X + \Psi_T^Y) \right] \times \left( \exp \left[ \rho_T^{W^X, W^Y} \int_0^T \sigma_s^X \sigma_s^Y ds + (B_T + C_T + D_T) \right] - 1 \right),$$

which is equivalent to:

$$Cov_T^{X,Y} = X_0 Y_0 \exp \left[ \int_0^T (E[r_s] + \theta_s^X) ds + \int_0^T (E[r_s] + \theta_s^Y) ds + (B_T + C_T + D_T) \right] \times \left( \exp \left[ \rho_T^{W^X, W^Y} \int_0^T \sigma_s^X \sigma_s^Y ds + (B_T + C_T + D_T) \right] - 1 \right)$$

■

Taking account of the interest rate randomness, the correlation  $\rho_T^{X,Y}$  of asset prices  $X$  and  $Y$  is given by:

$$\rho_T^{X,Y} = Cov_T^{X,Y} / (\sqrt{variance(X_T)} \sqrt{variance(Y_T)}),$$

which yields to:

$$\rho_T^{X,Y} = \frac{\exp[(B_T + C_T + D_T)] \left( \exp \left[ \rho_T^{W^X, W^Y} \int_0^T \sigma_s^X \sigma_s^Y ds + (B_T + C_T + D_T) \right] - 1 \right)}{\sqrt{\left( \exp \left[ \int_0^T [\sigma_s^X]^2 ds \right] - 1 \right)} \sqrt{\left( \exp \left[ \int_0^T [\sigma_s^Y]^2 ds \right] - 1 \right)}}.$$

Thus, we get:

$$\rho_T^{W^X, W^Y} = \frac{1}{\int_0^T (\sigma_s^X \sigma_s^Y) ds} \times \left( \text{Log} \left[ \frac{1 + \rho_T^{X,Y} \exp[-(B_T + C_T + D_T)]}{\sqrt{\left( \exp \left[ \int_0^T [\sigma_s^X]^2 ds \right] - 1 \right)} \sqrt{\left( \exp \left[ \int_0^T [\sigma_s^Y]^2 ds \right] - 1 \right)}} \right] \right).$$

Note that, if  $r_t$  were deterministic, then we would have a special case with  $\mu_s^X = r_s + \theta_s^X$ ,  $\mu_s^Y = r_s + \theta_s^Y$  and  $B_t = C_t = D_t = 0$ .

In such a case, the covariance of asset prices  $X$  and  $Y$  is given by:

$$\sigma_T^{X,Y} = X_0 Y_0 \exp \left[ \int_0^T (\mu_s^X + \mu_s^Y) ds \right] \left( \exp \left[ \rho_T^{W^X, W^Y} \int_0^T \sigma_s^X \sigma_s^Y ds \right] - 1 \right)$$

Therefore, for a fixed management period  $[0, T]$  and for a given correlation function  $\rho_T^{X,Y}$  defined by:

$$\rho_T^{X,Y} = \text{Cov}_T^{X,Y} / (\sqrt{\text{variance}(X_T)} \sqrt{\text{variance}(Y_T)}),$$

we find:

$$\rho_T^{X,Y} = \frac{\left( \exp \left[ \rho_T^{W^X, W^Y} \int_0^T \sigma_s^X \sigma_s^Y ds \right] - 1 \right)}{\sqrt{\left( \exp \left[ \int_0^T [\sigma_s^X]^2 ds \right] - 1 \right)} \sqrt{\left( \exp \left[ \int_0^T [\sigma_s^Y]^2 ds \right] - 1 \right)}}.$$

Thus, we get:

$$\rho_T^{W^X, W^Y} = \frac{1}{\int_0^T \sigma_s^X \sigma_s^Y ds} \times$$

$$\text{Log} \left[ 1 + \rho_T^{X,Y} \sqrt{\left( \exp \left[ \int_0^T [\sigma_s^X]^2 ds \right] - 1 \right)} \sqrt{\left( \exp \left[ \int_0^T [\sigma_s^Y]^2 ds \right] - 1 \right)} \right].$$

The previous formula can be applied not only to the real estate-stock case but also when the bond is involved. However, as detailed in Section 2, the bond modelling is such that the term structure is affine (see Relation 7).